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KEY

TO

ELEMENTS OF ALGEBRA;

DESIGNED FOR THE USE OF

CANADIAN GRAMMAR AND COMMON SCHOOLS,

CONTAINING

FULL SOLUTIONS TO NEARLY ALL THE PROBLEMS,

TOGETHER WITH

NUMEROUS EXPLANATORY REMARKS.

BY JOHN HERBERT SANGSTER, M.A., M.D.,

MATHEMATICAL MASTER AND LECTURER IN CHEMISTRY AND NATURAL PHILOSOPHY IN THE NORMAL SCHOOL FOR UPPER CANADA.

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PREFACE.

The following pages contain solutions to all, or nearly all the problems and exercises given in the Author's Elements of Algebra. In many cases, two or more solutions of the same problem are offered, so as to afford the student additional illustrations of the best and neatest modes of working; and of the application of artifices employed by the experienced algebraist in order to obtain a required result. On this account, also nearly every operation has been given at full length.

The Author hopes that the KEY will prove serviceable to the many who are privately prosecuting the study of Algebra, or endeavouring, without the aid of a living teacher, to prepare themselves for entrance into our Universities; and that it may likewise be of advantage to those teachers whose school duties are so many and varied as to render them unable to devote to the subject that time and study which long and intricate algebraic solutions in general require.

TORONTO, October, 1864.

CONTENTS.

KEY

TO

ELEMENTS OF ALGEBRA.

Exercise IV.

1. $1^3 - 1 = 1 - 1 = 0$
2. $3^3 - 3 \times 3 = 27 - 9 = 18$
3. $1 \times 2 + 3 \times 4 = 2 + 12 = 14$
4. $= 1^2 \times 2^2 - (3 - 1) = 1 \times 4 - 2 = 4 - 2 = 2$
5. $\sqrt{2 + 3 + 4} = \sqrt{9} = 3$
6. $0 \therefore m = 0$
7. $6 \times (1 \sim 3^2) = 6 \times (9 - 1) = 6 \times 8 = 48$
8. $(2^2 \times 4^2 - 3 \times 0)^{\frac{2}{3}} = (4 \times 16)^{\frac{2}{3}} = (\sqrt[3]{64})^2 = 4^2 = 16$
9. $(1 + 2) \times (4 - 0)^2 = 3 \times 4^2 = 3 \times 16 = 48$
10. $4\,\{1 - (4 - 3)\}^{\frac{2}{7}} = 4\,(1 - 1)^{\frac{2}{7}} = 4 \times 0^{\frac{2}{7}} = 4 \times 0 = 0$
11. $bcd = 2 \times 3 \times 4 = 24$
12. $(4^2 - 2 \times 3)^2\,(3^3 - 2 \times 3 \times 4)^3 = (16 - 6)^2\,(27 - 24)^3$ $= 10^2 \times 3^3 = 2700$
13. $\frac{1 + 1}{1 + 1} = \frac{2}{2} = 1 = a$; $\frac{b + 1}{\frac{1}{b} + 1} = \frac{2 + 1}{\frac{1}{2} + 1} = \frac{3}{\frac{1 + 2}{2}} = \frac{3}{\frac{3}{2}} = \frac{6}{3} = 2 = b$, &c.
14. $14 \times 1 - (3 \times 2 + 3) = 14 - 9 = 5$; $4^2 - 2\,(2 + 3) = 16 - 10 = 6$, and $5 < 6$

15. Each = 0 $\because$ m, one factor of each, is equal 0

16. $\sqrt{1 \times 4 \times 27 - 4\ (2+4)\ 3} = \sqrt{108 - 4 \times 6 \times 3} = \sqrt{36} = 6$ · $\sqrt[3]{(2+3) \times (16+9)} = \sqrt[3]{5 \times 25} = \sqrt[3]{125} = 5$, and $6 > 5$

17. $\dfrac{1 \times 4 \times 27 - 2 \times 4}{1+2+3+4} = \dfrac{108-8}{10} = \dfrac{100}{10} = 10$, and $2 \times (2+3) + 0$ $= 2 \times 5 = 10$

18. $\dfrac{1 \times 9 + 0 - (4-3)^2}{\sqrt[3]{2\ (16+9) + 2\ (3+4)}} = \dfrac{9 - 1^2}{\sqrt[3]{2 \times 25 + 2 \times 7}} = \dfrac{8}{\sqrt[3]{64}} = \dfrac{8}{4} = 2$; and $\{4 \times 3 - (4+3+2+1)\} = 12 - 10 = 2$

19. $(2-2)(3+8-3) + \{2 + (12-6)\} - 4\ (6-6) - \{18 - (9+1)\} + \{8 - (3+4) \times 1\}\ 2$; $= 0 \times 8 + (2+6) - 4 \times 0 - (18-10) + (8-7) \times 2$; $= 0 + 8 - 0 - 8 + 1 \times 2 = 1 \times 2 = 2$

20. $(9-1)(4-0) + 0 + 3\ \{1 + 3\ (4-3)\} = 8 \times 4 + 0 + 3\ (1+3)$ $= 32 + 3 \times 4 = 32 + 12 = 44$.

21. $\{(1-2) + (3+4)\}^2 + \{(3+0) - (2-1)\}^3 - \{(0+4) + (4-3)\}^2$ $= (-1+7)^2 + (3-1)^3 - (4+1)^2$; $= 6^2 + 2^3 - 5^2 = 36 + 8 - 25 = 19$

22. $\sqrt{(1+3) \times 4} + \sqrt[3]{9 \times (1+2)} + \{2\ (4+6)^2 + (28-12)\}^{\frac{1}{3}} - (24+1)^{\frac{3}{2}}$ $= \sqrt{4 \times 4} + \sqrt[3]{9 \times 3} + (2 \times 10^2 + 16)^{\frac{1}{3}} - (25)^{\frac{3}{2}} = \sqrt{16} + \sqrt[3]{27} + \sqrt[3]{216}$ $- (\sqrt{25})^3 = 4 + 3 + 6 - 5^3 = 13 - 125 = -112$

23. $\dfrac{7 \times \sqrt{0} + 3\sqrt{4} - (8+12)}{\frac{1}{3} \times 6 + 0^{\frac{2}{7}}} + \dfrac{36 - 28 + \{64 \times (1+3)\}^{\frac{1}{2}}}{\{(2-1)+1\}\{4-(2+0)\}} - \sqrt[3]{24-16}$

$= \dfrac{0+6-20}{2+0} + \dfrac{36 - 28 + \sqrt{256}}{2 \times 2} - \sqrt[3]{8} = -\dfrac{14}{2} + \dfrac{24}{4} - 2 = -7 + 6 - 2 = -3$

24. $\frac{1}{3}\{2(1+2)\} - \frac{1}{4}\ \{6(3+1)\} + \frac{1}{5}\ \{(3-2)(2+3)\} + \frac{1}{7}\ \{(4+3)$ $(1+6-6+4)^2\} = \frac{1}{3}(2 \times 3) - \frac{1}{4}\ (6 \times 4) + \frac{1}{5}\ (1 \times 5) + \frac{1}{7}\ (7 \times 25)$ $= 2 - 6 + 1 + 25 = 22$

25. $\dfrac{3(1+2-3)^3 + 11\{(3+6)(2-2+2)\}}{\{(9+2) - \sqrt{4}\}\ (4+3+4-0)} + \dfrac{\{(1+12)^2 - (27+10) - (3+4)\}^{\frac{2}{3}}}{0 + \sqrt{36} - 1}$

$+ \dfrac{(4+12-8)(4+3)}{7\ (4+4)} = \dfrac{3 \times 0^3 + 11\ (9 \times 2)}{(11-2)(11-0)} + \dfrac{(169 - 37 - 7)^{\frac{2}{3}}}{6-1} + \dfrac{8 \times 7}{7 \times 8}$

$= \dfrac{11 \times 18}{9 \times 11} + \dfrac{(\sqrt[3]{125})^2}{5} + \dfrac{56}{56} = 2 + \dfrac{5^2}{5} + 1 = 2 + 5 + 1 = 8$

Exercise IX.

1. $a + m - c + 6 + 5 - m - a - e + c + 3 - 5c - m = 14 - m - 5c - e$

2. $a - b - c - b + c + a - c + b + a - a - b - c = 2a - 2b - 2c$

3. $3a - 4 - 6y + x - 5a + 4 + 6y - 3a + 4 - 6 = x - 5a - 2$

4. $6 + (- \{ - (- \{ - (m) \}) \}) = 6 - \{ - (- \{ - (m) \}) \}$
$- 6 + (- \{ - (m) \}) = 6 - \{ - (m) \} = 6 + (m) = 6 + m$

5. $2a - 3c + 4d - 5d + (m + 3a) + 5a - (- 4 - d) - 3a + (4a - 5d - 4)$
$= 2a - 3c + 4d - 5d + m + 3a + 5a + 4 + d - 3a + 4a - 5d - 4$
$= 11a - 3c - 5d + m$

6. $m^2 - c^2 + a^2 + m^2 - 2a^2 + 2a^2 - m^2 - 5m^2 + c^2 + a^2 - c^2 + 3m^2$
$= 2a^2 - m^2 - c^2$

7. $1 + 1 - 1 + 1 + 1 - 1 = 2$

8. $a^2 + 2x - a^2 + a^2 + 2x^2 - 2m^2 + m^2 + a^2 + 2x + m^2 + 3a^2 + 3x + 3m^2$
$= 5a^2 + 7x + 2x^2 + 3m^2.$

9. $a^2bc + 3c^2 + 3a^2bc - m - c + 4a^2bc + c - 3c^2 - m = 8a^2bc - 2m$

10. $3a - 2a - 1 + a - 2 + a + 1 - a - 2a + 2 + a + 1 = a + 1$

11. $- a - b - c + a - c - c + a + 2a - 3b - 2c - 3b - a - b - c - a$
$= a - 8b - 6c$

12. $am - c - 7 + 5 - 7am + c + 3a + 5am - 4am - 6 + c - 9 - 3c - 4a$
$= - a - 5am - 2c - 17.$

Exercise XII.

1. $3am - 3x + 3y + 5ax + 15ay + 2am - 2my + 4ax + 4x$
$= 5am + x + 3y + 9ax + 15ay - 2my = 5am + x + 9ax + 3y + 15ay - 2my$
$= 5am + (1 + 9a)x + (3 + 15a - 2m)y$

2. $am - mx + my + 3mx + 3ax + 4a - 4y + 3ay + 3xy = am + 2mx$
$+ my + 3ax + 4a - 4y + 3ay + 3xy = 4a + am + 2mx + 3ax + 3xy$
$- 4y + my + 3ay = (4 + m) a + (2m + 3a) x + (3x - 4 + m + 3a) y$

3. $7a + 7b - 7c - 5b - 5x + 5bc - 3m + 3a + 3c$
$= 10a + 2b - 4c - 5x - 5bc - 3m = 10a - 5x - 5bc + 2b - 4c - 3m$
$= 5 (2a - x - bc) + 2 (b - 2c) - 3m$

4. $ax + mx - 3amxy - 3cxy + 2ay^2 - 2cmy^2 + ax + ay^2 + cxy + axy - by^2 - fy^2$
$= 2ax + mx - 3amxy - 2cxy + 3ay^2 - 2cmy^2 + axy - by^2 - fy^2$
$= 2ax + mx - 3amxy - 2cxy + axy + 3ay^2 - 2cmy^2 - by^2 - fy^2$
$= (2a + m)\ x - (3am + 2c - a)\ xy + (3a - 2cm - b - f)\ y^2$

5. $3ay - 3by + 3cy - 2mx + cx - 3amx - 3amy + 3amz - (3amx + 3amy + 3amz + 2cx + 2cz + acy - acz) = 3ay + 3by + 3cy - 2mx + cx - 3amx - 3amy + 3amz - 3amx - 3amy - 3amz - 2cx - 2cz - acy + acz$
$= 3ay - 3by + 3cy - 2mx + cx - 6amx - 6amy - 2cz - acy + acz$
$= 3ay - 3by + 3cy - 6amy - acy - 2mx + cx - 6amx - 2cz + acz$
$= (3a - 3b + 3c - 6am - ac)\ y - (2m - c + 6am)\ x - (2 - a)\ cz$

6. $11amy + 11bmy - 3axy + 3bxy - 3cxy - (2acp + 2acxy - 3cm + 6cxy - 3cy^2 - 3ay - 3ac)$
$= 11amy + 11bmy - 3axy + 3bxy - 3cxy - 2acp - 2acxy + 3cm - 6cxy + 3cy^2 + 3ay + 3ac$
$= 11amy + 11bmy + 3cy^2 + 3ay - 3axy + 3bxy - 2acxy - 9cxy - 2acp + 3cm + 3ac$
$= \{11\ (a + b)m + 3\ (cy + a)\}\ y - \{3(a - b) + (2a + 9)c\}\ xy + 3\ (m + a)\ c - 2acp$

Exercise XVIII.

1. $\{(a - b) + c\}\{(a - b) - c\} = (a - b)^2 - c^2 =$ &c.
$\{a - (b - c)\}\{a + (b - c)\} = a^2 - (b - c)^2 =$ &c.
$\{a + (b + c)\}\{a - (b + c)\} = a^2 - (b + c)^2 =$ &c.

2. $\{4 + (3a - 2c)\}\{4 - (3a - 2c)\} = 16 - (3a - 2c)^2 =$ &c.
$\{2a - (x - 3m^2)\}\{2a + (x - 3m^2)\} = 4a^2 - (x - 3m^2)^2 =$ &c.
$\{2xy + (2a - 3y)\}\{2xy - (2a - 3y)\} = 4x^2y^2 - (2a - 3y)^2 =$ &c.

3. $\{(2a - 3c) + (2x - 3y)\}\ \{(2a - 3c) - (2x - 3y)\} = (2a - 3c)^2 - (2x - 3y)^2 =$ &c.
$\{(a + 3d) + (2c + 4m)\}\{(a + 3d) - (2c + 4m)\} = (a + 3d)^2 - (2c + 4m)^2 =$ &c.

4. $\{(3a - m^2) - (2 - xy)\}\{(3a - m^2) + (2 - xy)\} = (3a - m^2)^2 - (2 - xy)^2 =$ &c.

$\{(2a^2 - 3x^2) + (1 + y^2)\}\{(2a^2 - 3x^2) - (1 + y^2)\} = (2a^2 - 3x^2)^2 - (1 + y^2)^2 =$ &c.

5. $(5ab + 6a^2 - 6b^2) - (4a^2 - 16ab + 16b^2) - 4(9 - a^2) - 4(4a^2 - 4ab + b^2) = 5ab + 6a^2 - 6b^2 - 4a^2 + 16ab - 16b^2 - 36 + 4a^2 - 16a^2 + 16ab - 4b^2 =$ &c.

6. $(24axy - 16a^2 - 9x^2y^2) + 3(4a^2 + 4axy + x^2y^2) - 7(x^2y^2 - 9a^2) + 4(4a^2 - 12axy + 9x^2y^2) = 24axy - 16a^2 - 9x^2y^2 + 12a^2 + 12axy + 3x^2y^2 - 7x^2y^2 + 63a^2 + 16a^2 - 48axy + 36x^2y^2 =$ &c.

7. $(1 - x^2)(1 + x^2)(1 + x^4) +$ &c. to 7 terms $= (1 - x^4)(1 + x^4)(1 + x^8) +$ &c. to 6 terms $= (1 - x^8)(1 + x^8)(1 + x^{16}) +$ &c. to 5 terms $= (1 - x^{16})(1 + x^{16})(1 + x^{32}) +$ &c. to 4 terms $- (1 - x^{32})(1 + x^{32})(1 + x^{64}) = (1 - x^{64})(1 + x^{64}) = 1 - x^{128}$

8. Product of first two terms $= a^2 - x^2y^2$; of first three terms $= a^4 - x^4y^4$; of first four terms $= a^8 - x^8y^8$, and so on.

Now the index of each term in the product of the first two factors $= 2 = 2^1 = 2^{2-1}$

Index of each term in the product of the first three factors $= 4 = 2^2 = 2^{3-1}$

Index of each term in the product of the first four factors $= 8 = 2^3 = 2^{4-1}$, and so on

Therefore the index of each term in the product of n such factors $= 2^{n-1}$ $\therefore$ $(a - xy)(a + xy)(a^2 + x^2y^2)$ .... to n terms $= a^{2^{n-1}} - (xy)^{2^{n-1}}$

EXERCISE XIX.

4. $(a^3 + b^3)(a^3 - b^3) =$ &c.

5. $(a^3)^3 - (x^3)^3 = (a^3 - x^3)(a^6 + a^3x^3 + x^6) =$ &c.

7. $(a^2 + m^2x^2)(a^2 - m^2x^2) =$ &c.

8. $(2a)^5 + x^5 = (2a + x)\{(2a)^4 - (2a)^3x + (2a)^2x^2 - 2ax^3 + x^4\}$ $= \&c.$

9. $3^4 - (2c)^4 = \{3^2 + (2c)^2\}\{3^2 - (2c)^2\} = (9 + 4c^2)(3 + 2c)(3 - 2c)$

10. $(3m)^5 - (2c)^5 = (3m - 2c)\{(3m)^4 + (3m)^3(2c) + (3m)^2(2c)^2$ $+ (3m)(2c)^3 + (2c)^4\} = \&c.$

11. $(a^7)^3 + (x^7)^3 - (a^7 + x^7)(a^{14} - a^7x^7 + x^{14}) = \&c.$

12. $(a^4)^5 + (m^4)^5 = (a^4 + m^4)(a^{16} - a^{12}m^4 + a^8m^8 - a^4m^{12} + m^{16})$

13. $(c^8)^3 + (x^8)^3 = \&c.$

14. $(x^{10})^3 + (m^{10})^3 = (x^{10} + m^{10})\ (x^{20} - x^{10}m^{10} + m^{20}) = \&c$

15. $(a^{24} + c^{24})(a^{12} + c^{12})(a^6 + c^6)(a^3 + c^3)(a^3 - c^3)$ $= \{(a^8)^3 + (c^8)^3\}\{(a^4)^3 + (c^4)^3\}\{(a^2)^3 + (c^2)^3\}(a^3 + c^3)(a^3 - c^3) = \&c.$

16. $(a^{32})^3 + (m^{32})^3 = \&c.$

17. $(a^{54} + c^{54})(a^{27} + c^{27})(a^{27} - c^{27}) = \{(a^{18})^3 + (c^{18})^3\}$ $\{(a^9)^3 + (c^9)^3\}\{(a^9)^3 - (c^9)^3\} = (a^{18} + c^{18})(a^{36} - a^{18}c^{18} + c^{36})$ $(a^9 + c^9)\ (a^{18} - a^9c^9 + c^{18})\ (a^9 - c^9)\ (a^{18} + a^9c^9 + c^{18})$ $= \{(a^6)^3 + (c^6)^3\}\{(a^3)^3 + (c^3)^3\}\{(a^3)^3 - (c^3)^3\}(a^{36} - a^{18}c^{18} + c^{36})$ $(a^{18} - a^9c^9 + c^{18})(a^{18} + a^9c^9 + c^{18}) = (a^6 + c^6)(a^{12} - a^6c^6 + c^{12})$ $(a^3 + c^3)(a^6 - a^3c^3 + c^6)(a^3 - c^3)(a^6 + a^3c^3 + c^6)(a^{36} - a^{18}c^{18}$ $+ c^{36})(a^{18} - a^9c^9 + c^{18})(a^{18} + a^9c^9 + c^{18}) = \{(a^2)^3 + (c^2)^3\}$ $(a^3 + c^3)\ (a^3 - c^3)\ (a^{36} - a^{18}c^{18} + c^{36})\ (a^{18} - a^9c^9 + c^{18})$ $(a^{18} + a^9c^9 + c^{18})(a^{12} - a^6c^6 + c^{12})(a^6 - a^3c^3 + c^6)(a^6 + a^3c^3$ $+ c^6) = \&c.$

18. $(m^{48})^3 + (c^{48})^3 = (m^{48} + c^{48})\ (m^{96} - m^{48}c^{48} + c^{96})$ $= \{(m^{16})^3 + (c^{16})^3\}(m^{96} - m^{48}c^{48} + c^{96}) = \&c.$

19. $(a^2)^7 + (m^2)^7 = \&c.$

20. $(a^{27}m^{27})^3 - (p^{27})^3 = (a^{27}m^{27} - p^{27})(a^{54}m^{54} + a^{27}m^{27}p^{27}$ $+ p^{54}) = \{(a^9m^9)^3 - (p^9)^3\}(a^{54}m^{54} + a^{27}m^{27}p^{27} + p^{54}) = (a^9m^9 - p^9)$ $(a^{18}m^{18} + a^9m^9p^9 + p^{18})(a^{54}m^{54} + a^{27}m^{27}p^{27} + p^{54}) = \{(a^3m^3)^3$ $- (p^3)^3\}(a^{18}m^{18} + a^9m^9p^9 + p^{18})(a^{54}m^{54} + a^{27}m^{27}p^{27} + p^{54}) = \&c.$

Exercise XX.

1. $a - x + x - a - a + a + a - x + a - x - a = a - 2x$

2. $3(a^2 - x^2) - 2(a^2 - 4ax + 4x^2) - (12ax - 9a^2 - 4x^2) - 4(9x^2 - a^2)$
$= 3a^2 - 3x^2 - 2a^2 + 8ax - 8x^2 - 12ax + 9a^2 + 4x^2 - 36x^2 + 4a^2$
$= 14a^2 - 43x^2 - 4ax$

(4)

$$\begin{array}{l}
a^m + x^{p+q} \\
\quad a^c - x^{m-p} \\
\hline
a^{c+m} + a^c x^{p+q} \\
\quad - a^m x^{m-p} - x^{m+q} \\
\hline
a^{c+m} + a^c x^{p+q} - a^m x^{m-p} - x^{m+q}
\end{array}$$

(5)

$$\begin{array}{l}
a + x)\, a^n - x^n\, (a^{n-1} - a^{n-2}x + a^{n-3}x^2 - \&c. \\
\quad a^n + a^{n-1}x \\
\hline
\quad -a^{n-1}x - x^n \\
\quad -a^{n-1}x - a^{n-2}x^2 \\
\hline
\quad\quad a^{n-2}x^2 - x^n \\
\quad\quad a^{n-2}x^2 + a^{n-3}x^3 \\
\hline
\quad\quad\quad -a^{n-3}x^3 - x^n, \&c.
\end{array}$$

(7)

$$\begin{array}{l}
1 - 1)\; 1\; (1 + 1 + 1 + 1, \&c. \\
\quad 1 - 1 \\
\hline
\quad\quad + 1 \\
\quad\quad 1 - 1 \\
\hline
\quad\quad\quad 1 \\
\quad\quad\quad 1 - 1 \\
\hline
\quad\quad\quad\quad 1, \&c.
\end{array}$$

8. $(a^9 + x^9)(a^9 - x^9) = \{(a^3)^3 + (x^3)^3\}\{(a^3)^3 - (x^3)^3\} = (a^3 + x^3)$
$(a^3 - x^3)(a^6 - a^3x^3 + x^6)(a^6 + a^3x^3 + x^6) = \&c.$

9. $x^2m^2(a^4x^2 - 4a^2xp + 4p^2) = m^2x^2(a^2x - 2p)^2.$

10. $\dfrac{\sqrt{4(6+3)}}{12-0} + \sqrt[3]{0} - \dfrac{\{(2\{3+4\} - 1)^2 + 6\} - \{12(3+4) + 1\}}{0 + \sqrt{4\,(12+1)} - 3 - (2+3+1)}$

$= \dfrac{\sqrt{36}}{12} - \dfrac{\{(14-1)^2 + 6\} - (84+1)}{\sqrt{52} - 3 - 6} = \dfrac{6}{12} - \dfrac{(169+6) - 85}{\sqrt{49} - 6}$

$= \dfrac{1}{2} - \dfrac{175 - 85}{7 - 6} = \dfrac{1}{2} - 90 = -89\frac{1}{2}$

(11)

$$\begin{array}{l}
1 + 2 + 3 + 2 + 1 \\
1 - 2 + 1 \\
\hline
1 + 2 + 3 + 2 + 1 \\
\quad - 2 - 4 - 6 - 4 - 2
\end{array}
\qquad\qquad
\begin{array}{l}
1 + 2 + 3 + 4. \\
1 - 2 - 3 \\
\hline
1 + 2 + 3 + 4 \\
\quad - 2 - 4 - 6 - 8
\end{array}$$

$$\begin{array}{l} \quad\quad\; 1+2+3+2+1 \\ \hline 1+0+0-2+0+0+1 \end{array} \qquad \begin{array}{l} \quad\;\; -3-6-9-12 \\ \hline 1+0-4-8-17-12 \end{array}$$

$x^6+0x^5+0x^4-2x^3+0x^2+0x+1$ $a^5+0a^4b-4a^3b^2-8a^2b^3-17ab^4-12b^5$

$=x^6-2x^3+1$ $=a^5-4a^3b^2-8a^2b^3-17ab^4-12b^5$

(12)

$$x^4+0x^3-(a^2-b+c)\,x^2+(ab+ac)\,x-bc \div x^2+ax-c$$

$$\begin{array}{r|llll} 1 & 1+0 & -(a^2-b+c) & +(ab+ac) & -bc \\ -a & \quad -a+ & a^2 & -ab & \\ +c & & +c & -ac & +bc \\ \hline & 1-a+ & b & & \end{array} \qquad = x^2-ax+b.$$

13. $(a^{64}-m^{64})=(a^{32}+m^{32})(a^{32}-m^{32})=(a^{32}+m^{32})(a^{16}+m^{16})$ $(a^{16}-m^{16})=$ &c.

14. $(a^2+c^2)\times 4^{\text{th}}$ factor $=a^{22}+c^{22}$; $(a-c)\times 5^{\text{th}}$ factor $=a^{11}-c^{11}$ and $(a+c)\times$ last factor $=a^{11}+c^{11}$. Hence required value $=(a^{11}-c^{11})(a^{11}+c^{11})(a^{22}+c^{22})=(a^{22}-c^{22})(a^{22}+c^{22})=a^{44}-c^{44}$

15. $a+b=0$ and $a=\frac{1}{2}$ $\therefore$ $b=-\frac{1}{2}$; $a+b+c=0$, and $a+b=0$ $\therefore$ $c=0$; $\frac{1}{4}\{\frac{1}{4}-(-\frac{1}{2})\frac{1}{2}\}=\frac{1}{4}\{\frac{1}{4}-(-\frac{1}{4})\}=\frac{1}{4}(\frac{1}{4}+\frac{1}{4})=\frac{1}{4}\times\frac{1}{2}=\frac{1}{8}$

16. $a^3-b^3-3a^2b+3ab^2+3a^2b+3ab^2+a^3+b^3=2a^3+6ab^2$ $=2a\,(a^2+3b^2)$

17. $3\,(a-m)^2-2\,(4a^2-9m^2)-(10m^2+6am)+7\,(a^2-m^2)+(10am-4m^2)=3a^2-6am+3m^2-8a^2+18m^2-10m^2-6am+7a^2-7m^2+10am-4m^2=2a^2-2am=2a\,(a-m)$

18. $=m\{(m-2a)(m-2b)+(m-2b)(m-2c)+(m-2c)(m-2a)\}$

$=m\{(b-a+c)(a-b+c)+(a-b+c)(a+b-c)+(a+b-c)(b-a+c)\}$

$=m\{c^2-(b-a)^2+a^2-(b-c)^2+b^2-(a-c)^2\}$

$=m(c^2-b^2+2ab-a^2+a^2-b^2+2bc-c^2+b^2-a^2+2ac-c^2)$

$=m(2ab+2ac+2bc-a^2-b^2-c^2)$

$=m(2ab+2ac+2bc-a^2-b^2-c^2-4ab+4ab)$

$=m(2ac-2ab+2bc-a^2-b^2-c^2+4ab)$

$=m\{(c-a-b)(a+b-c)+4ab\}=m\{(c-a-b)(a+b+c-2c)+4ab\}$

$=m\{(c-a-b)(m-2c)+4ab\}=m\{m(c-a-b)-2c(c-a-b)+4ab\}$

$=8abc+m^2(c-a-b)+4abm-2cm(c-a-b)-8abc$

$= 8abc + m^2(m - 2a - 2b) + 4abm - 2cm(m - 2a - 2b) - 8abc$
$= 8abc + m(m^2 - 2am - 2bm + 4ab) - 2c(m^2 - 2am - 2bm + 4ab)$
$= 8abc + (m - 2c)(m^2 - 2am - 2bm + 4ab)$
$= 8abc + (m - 2c)(m - 2b)(m - 2a)$

Exercise XXI.

1. $3b \times 6ab^2m$, and $4am^2 \times 6ab^2m$.

2. $3a^2m^2 \times 7a^2$, $3a^2m^2 \times 6am$, and $3a^2m^2 \times 5m^2$.

3. $axy(8ax + 17m - 3am^2x)$, and $xy(5 + 3a - 14a^2x)$.

4. $(x^2 - mx^2) + (2x - 2mx)$, and $(x^2 + 4x + 4) + (ax + 2a)$; that is of $x^2(1 - m) + 2x(1 - m)$; and $(x + 2)^2 + a(x + 2)$; that is of $(x^2 + 2x)(1 - m)$, and $(x + 2)(x + 2 + a)$; that is of $x(x + 2)(1 - m)$, and $(x + 2)(x + 2 + a)$.

5. That is of $3a^2(a - x)(a + x)$, and $4a^2x^2(a - x)^2$;

6. That is of $3m^3(a^3 - m^3)(a + m)$; $4m^3(a^2 - m^2)^2$, and $4m^2(a^2 - m^2)(a - m)$; that is of $3m^3(a^2 - m^2)(a^2 + am + m^2)$; $4m^3(a^2 - m^2)^2$, and $4m^2(a^2 - m^2)(a - m)$

7. That is of $(x - 7)(x + 3)$; $(x - 7)(x - 5)$, and $(x - 7)(x + 12)$

8. That is of $a^2(x - 1)^2$, and $a^2(x - 1)(x - 2)$

9. That is of $(x + 4)(x - 1)$; $(x - 1)^2$, and $(x - 1)(x + 1)$

Exercise XXII.

(1)

$x^2 - x - 6)x^2 - 5x - 14(1$
$x^2 - x - 6$
$- 4x - 8$
$- 4(x + 2)$
$x + 2)x^2 - x - 6(x - 3$
$x^2 + 2x$
$- 3x - 6$
$- 3x - 6$

(2)

$2x^3 - 12x^2 + 21x - 10)x^4 - 8x^3 + 21x^2 - 20x + 4$
2
$2x^4 - 16x^3 + 42x^2 - 40x + 8(x - 2$
$2x^4 - 12x^3 + 21x^2 - 10x$
$- 4x^3 + 21x^2 - 30x + 8$
$- 4x^3 + 24x^2 - 42x + 20$
$- 3x^2 + 12x - 12$
$- 3(x^2 - 4x + 4)$

$$x^2 - 4x + 4)\,2x^3 - 12x^2 + 21x - 10\,(2x - 4$$
$$2x^3 - 8x^2 + 8x$$
$$-4x^2 + 13x - 10$$
$$-4x^2 + 16x - 16$$
$$-3x + 6$$
$$-3(x - 2)$$

$$x - 2)\,x^2 - 4x + 4\,(x - 2$$
$$x^2 - 2x$$
$$-2x + 4$$
$$-2x + 4$$

3. $(a^2 - ax) - (7a - 7x)$, and $(a^3 - a^2x) - (3a - 3x)$

$a(a - x) - 7(a - x)$, and $a^2(a - x) - 3(a - x)$

$(a - 7)(a - x)$, and $(a^2 - 3)(a - x)$

4. $x(x^2 + x - 12)$, and $x^2(x + 4) + 5(x + 4)$

$x(x + 4)(x - 3)$, and $(x^2 + 5)(x + 4)$.

5. $a^2 - ab - 2b^2)\,a^2 - 3ab + 2b^2\,(1$
$$a^2 - ab - 2b^2$$
$$-2ab + 4b^2$$
$$-2b(a - 2b)$$

$$a - 2b)\,a^2 - ab - 2b^2\,(a + b$$
$$a^2 - 2ab$$
$$ab - 2b^2$$
$$ab - 2b^2$$

6. $a^2 - 5ab + 4b^2)\,a^3 - a^2b + 3ab^2 - 3b^3\,(a + 4b$
$$a^3 - 5a^2b + 4ab^2$$
$$4a^2b - ab^2 - 3b^3$$
$$4a^2b - 20ab^2 + 16b^3$$
$$19ab^2 - 19b^3$$
$$19b^2(a - b)$$

$$a - b)\,a^2 - 5ab + 4b^2\,(a - 4b$$
$$a^2 - ab$$
$$-4ab + 4b^2$$
$$-4ab + 4b^2$$

7. Rejecting the factor 2 from the first quantity

$$15x^4-9x^3+47x^2-21x+28)60x^6-36x^5+48x^4-45x^3+42x^2-45x+12(4x^2-28$$

$$\underline{60x^6-36x^5+188x^4-84x^3+112x^2}$$

$$-140x^4+39x^3-70x^2-45x+12$$

$$\underline{\qquad 3 \qquad}$$

$$-420x^4+117x^3-210x^2-135x+36$$

$$\underline{-420x^4+252x^3-1316x^2+588x-784}$$

$$-135x^3+1106x^2-723x+820$$

$$135x^3-1106x^2+723x-820)15x^4-9x^3+47x^2-21x+28(x+205$$

$$\underline{\qquad 9 \qquad}$$

$$135x^4-81x^3+423x^2-189x+252$$

$$\underline{135x^4-1106x^3+723x^2-820x}$$

$$1025x^3-300x^2+631x+252$$

$$\underline{\qquad 27 \qquad}$$

$$27675x^3-8100x^2+17037x+6804$$

$$\underline{27675x^3-226730x^2+148215x-168100}$$

$$218630x^2-131178x+174904$$

$$43726(5x^2-3x+4)$$

$$5x^2-3x+4)135x^3-1106x^2+723x-820(27x-205$$

$$\underline{135x^3-\quad 81x^2+108x}$$

$$-1025x^2+615x-820$$

$$\underline{-1025x^2+615x-820}$$

8. That is of $2b(3a^3-3a^2y-y^3+ay^2)$, and $3b(4a^2+y^2-5ay)$

That is of $2b\{(3a^3-3a^2y)+(ay^2-y^3)\}$, and $3b\{(4a^2-4ay)-(ay-y^2)\}$

$2b\{3a^2(a-y)+y^2(a-y)\}$, and $3b\{4a(a-y)-y(a-y)\}$

$2b(a-y)(3a^2+y^2)$, and $3b(a-y)(4a-y)$;

Otherwise,

$$4a^2-5ay+y^2)\,3a^3-3a^2y+ay^2-y^3$$

$$\underline{\qquad 4 \qquad}$$

$$12a^3-12a^2y+4ay^2-4y^3\,(3a+3y$$

$$\underline{12a^3-15a^2y+3ay^2}$$

$$3a^2y+\ ay^2-4y^3$$

$$\underline{\qquad 4 \qquad}$$

$$12a^2y+4ay^2-16y^3$$

$$\begin{array}{r} 12a^2y - 15ay^2 + 3y^3 \\ \hline 19ay^2 - 19y^3 \\ 19y^2(a-y) \end{array}$$

$$a - y)\ 4a^2 - 5ay + y^2\ (4a - y$$
$$\begin{array}{r} 4a^2 - 4ay \\ \hline -ay + y^2 \\ -ay + y^2 \\ \hline \end{array} \quad \therefore\ G.\ C.\ M. = b\,(a-y)$$

9. $a^2 + 12a - 28)\ a^3 + 9a^2 + 27a - 98\ (a - 3$
$$\begin{array}{r} a^3 + 12a^2 - 28a \\ \hline -3a^2 + 55a - 98 \\ -3a^2 - 36a + 84 \\ \hline 91a - 182 \\ 91\,(a-2) \end{array}$$

$$a - 2)\ a^2 + 12a - 28(a + 14$$
$$\begin{array}{r} a^2 - 2a \\ \hline 14a - 28 \\ 14a - 28 \\ \hline \end{array}$$

10. $8b^2\,(a^3 - 3a^2b + 3ab^2 - b^3)$, and $12a^2\,(a^2 - 2ab + b^2)$

That is of $8b^2\,(a-b)^3$, and $12a^2\,(a-b)^2$

11. Rejecting the factor 2 from the first quantity and multiplying the second by 3

$$3a^5 + 10a^4 - 6a^3 - 24a^2 + 11a + 6)\,3a^6 + 12a^5 - 9a^4 - 48a^3 + 33a^2 + 36a - 27\ (a + 2$$
$$\begin{array}{r} 3a^6 + 10a^5 - 6a^4 - 24a^3 + 11a^2 + 6a \\ \hline 2a^5 - 3a^4 - 24a^3 + 22a^2 + 30a - 27 \\ 3 \\ \hline 6a^5 - 9a^4 - 72a^3 + 66a^2 + 90a - 81 \\ 6a^5 + 20a^4 - 12a^3 - 48a^2 + 22a + 12 \\ \hline -29a^4 - 60a^3 + 114a^2 + 68a - 93 \end{array}$$

$$29a^4 + 60a^3 - 114a^2 - 68a + 93)\,3a^5 + 10a^4 - 6a^3 - 24a^2 + 11a + 6\,(3a + 110$$
$$\begin{array}{r} 29 \\ \hline 87a^5 + 290a^4 - 174a^3 - 696a^2 + 319a + 174 \\ 87a^5 + 180a^4 - 342a^3 - 204a^2 + 279a \\ \hline 110a^4 + 168a^3 - 492a^2 + 40a + 174 \end{array} \times 29 =$$

$$3190a^4 + 4872a^3 - 14268a^2 + 1160a + 5046$$
$$3190a^4 + 6600a^3 - 12540a^2 - 7480a + 10230$$
$$-1728a^3 - 1728a^2 + 8640a - 5184$$
$$-1728\,(a^3 + a^2 - 5a + 3)$$
$$a^3 + a^2 - 5a + 3)\ 29a^4 + 60a^3 - 114a^2 - 68a + 93\ (29a + 31$$
$$29a^4 + 29a^3 - 145a^2 + 87a$$
$$31a^3 + 31a^2 - 155a + 93$$
$$31a^3 + 31a^2 - 155a + 93$$

12. Rejecting the factor 2 from the first, and $3c$ from the second

$$-a^2b - 8ab^2 + 6b^3)\ a^4 - 3a^3b - 8a^2b^2 + 18ab^3 - 8b^4\ (a - 2b$$
$$a^4 - a^3b - 8a^2b^2 + 6ab^3$$
$$-2a^3b + 12ab^3 - 8b^4$$
$$-2a^3b + 2a^2b^2 + 16ab^3 - 12b^4$$
$$-2a^2b^2 - 4ab^3 + 4b^4$$
$$-2b^2\,(a^2 + 2ab - 2b^2)$$
$$a^2 + 2ab - 2b^2)\ a^3 - a^2b - 8ab^2 + 6b^3\ (a - 3b$$
$$a^3 + 2a^2b - 2ab^2$$
$$-3a^2b - 6ab^2 + 6b^3$$
$$-3a^2b - 6ab^2 + 6b^3$$

EXERCISE XXIII.

1. $4 \times -3 \times a^2b^2x^2y^2 = -12a^2b^2x^2y^2$

2. $4 \times 3 \times a^2x^2y^2z^2 = 12a^2x^2y^2z^2$

3. $(x - y)^2\,(x^2 - y)^2 = \{(x - y)(x^2 - y)\}^2 = (x^3 - x^2y - xy + y^2)^2$

4. $(x^2 + xy + y^2)\,(x^4 - y^4) = x^6 + x^5y + x^4y^2 - x^2y^4 - xy^5 - y^6$

5. $x^2(1 - x)^2$; $(x - 1)(x + 1)$, and $4x(1 + x)$ that is $x^2(1 - x)^2$; $(1 - x)(1 + x)$, and $4x(1 + x) = 4x^2(1 - x)^2\,(1 + x) = 4x^5 - 4x^4 - 4x^3 + 4x^2$

6. $a^6 - b^6$ contains $a^3 - b^3$, and $a^3 + b^3$ as factors; $\therefore$ *l. c. m*
$= 36(a - b)(a^6 - b^6) = 36(a^7 - a^6b - ab^6 + b^7) = 36a^7 - 36a^6b - 36ab^6 + 36b^7$

7. $x(x - 3)$; $(x - 3)(x - 7)$; and $x(x - 7)$

$\therefore$ *l. c. m.* $= x(x^2 - 10x + 21) = x^3 - 10x^2 + 21x$

8. $(a^3 - x^3)$, and $(a^2 - ax) - (a - x)$

$a^3 - x^3$, and $a(a - x) - (a - x)$

$a^3 - x^3$, and $(a - x)(a - 1)$

$\therefore$ *l. c. m* $= (a^3 - x^3)(a - 1) = a^4 - a^3 - ax^3 + x^3$

9 *G. C. M.* of two given quantities is $a^2 - 7a + 12$

$$\frac{a^3 - 9a^2 + 26a - 24}{a^2 - 7a + 12} = a - 2$$

$(a^3 - 8a^2 + 19a - 12)(a - 2) = a^4 - 10a^3 + 35a^2 - 50a + 24$

10. $3(a - b)(a^2 + ab + b^2)$; $4(a - b)^3$; $5(a - b)(a + b)(a^2 + b^2)$ $6(a - b)^2$, and $\{(a - b)(a + b)\}^3$

Or of $3(a - b)(a^2 + ab + b^2)$; $4(a - b)^3$; $5(a - b)(a + b)(a^2 + b^2)$; $6(a - b)^2$, and $(a + b)^3(a - b)^3$

$\therefore$ *l. c. m.* $= 3 \times 4 \times 5(a - b)^3(a + b)^3(a^2 + ab + b^2)(a^2 + b^2)$
$= 60(a^{10} + a^9b - a^8b^2 - 2a^7b^3 - 2a^6b^4 + 2a^4b^6 + 2a^3b^7 + a^2b^8 - ab^9 - b^{10})$

Exercise XXIV.

1. $\dfrac{a(a - b)}{a(x + y)}$

2. $\dfrac{m(2a + mx - m^2)}{m(3a^2 + m)}$

3. $\dfrac{c(1 + a)}{n(1 + a)}$

4. $\dfrac{a^2b(1 + b + m)}{x(1 + b + m)}$

5. $\dfrac{abc^2}{b(a + c)}$

6. $\dfrac{ax^2y^3}{x(a^2xm + ay + x^2y^2z^3)}$

7. $\dfrac{7x^2y^2(3 - 5x)}{14x^3y^2}$

8. $\dfrac{a - m}{(a - m)(a + m)}$

9. $\dfrac{(a + b)(a^2 - ab + b^2)}{(a + b)(a - b)}$

10. $\dfrac{(a - b)(a - b)}{(a - b)(a^2 + ab + b^2)}$

11. $\dfrac{(a+b)(a^2-ab+b^2)}{(a-b)(a^2+ab+b^2)}$ 12. $\dfrac{(a^2-m^2)(a^4+a^2m^2+m^4)}{a^2-m^2}$

13. $\dfrac{(a^2-m^2)(a^2+m^2)}{a^3(a^2-m^2)}$ 14. $\dfrac{7(x^2-3x+5)}{11(x^2-3x+5)}$.

15. $\dfrac{(x-7)(x-4)}{(x-7)(x+3)}$ 16. $\dfrac{(2x+3)(2x+3)}{(2x+3)(x-4)}$

17. $\dfrac{x^2(x+2y+3y^2)}{x^2(2x^2-3xy-5y^2)}$ 18. $\dfrac{(a^2-ab+b^2)(a-b)}{(a^2-ab+b^2)(a^2+ab+b^2)}$.

19. $\dfrac{(a^2-m^2)(a^2+m^2)}{a^2(a-m)-m^2(a-m)}=\dfrac{(a^2-m^2)(a^2+m^2)}{(a^2-m^2)(a-m)}$.

20. $\dfrac{(ac+bc)+(ad+bd)}{(am+bm)+(2ap+2bp)}=\dfrac{c(a+b)+d(a+b)}{m(a+b)+2p(a+b)}$

$=\dfrac{(c+d)(a+b)}{(m+2p)(a+b)}$ 21. $\dfrac{(x+a)(x+b)}{(x+c)(x+b)}$

22. $\dfrac{(x-1)(2x^2+3x-5)}{(x-1)(7x-5)}$ 23. $\dfrac{(a+m)(a^2+2am+m^2-x^2)}{\{x^2-(a-m)^2\}\{(a+m)^2-x^2\}}$

$=\dfrac{(a+m)(a^2+2am+m^2-x^2)}{(a^2+2am+m^2-x^2)(x^2-a^2+2am-m^2)}$

24. $\dfrac{(a^4+x^4)(a^8-a^4x^4+x^8)}{(a^4+x^4)(a^{16}-a^{12}x^4+a^8x^8-a^4x^{12}+x^{16})}$

Exercise XXV.

2. $\dfrac{a^3+a^2+a-a^2-a-1+2}{a-1}=\dfrac{a^3+1}{a-1}$

3. $\dfrac{3ax+9a-yx-3y-(3a^2-30)}{x+3}=\dfrac{3ax+9a-xy-3y-3a^2+30}{x+3}$

4. $\dfrac{3ax-3ay+xy-y^2-2a-xy}{x-y}=\dfrac{3ax-3ay-2a-y^2}{x-y}$

5. $\dfrac{3a^2x+3ax^2-ay^2-xy^2+am+mx-3ax^2-xy^2}{a+x}$

$=\dfrac{3a^2x-ay^2-2xy^2+am+mx}{a+x}$

6. $\dfrac{xyz + 2mxy + mz^2 + 2m^2z + xyz - z^2m - 2m^2z}{z + 2m} = \dfrac{2xyz + 2mxy}{z + 2m}$

$= \dfrac{2xy(z + m)}{z + 2m}$

7. $\dfrac{(a + b)^3 - (a - b)^3}{a + b}$

$= \dfrac{a^3 + 3a^2b + 3ab^2 + b^3 - (a^3 - 3a^2b + 3ab^2 - b^3)}{a + b} = \dfrac{6a^2b + 2b^3}{a + b} = \dfrac{2b(3a^2 + b^2)}{a + b}$

8. $\dfrac{a^2 + m^2 - a^2 + m^2}{a^2 + m^2} = \dfrac{2m^2}{a^2 + m^2}$

9. $\dfrac{a^2 + x^2 - a^2 + 2ax - x^2}{a^2 + x^2} = \dfrac{2ax}{a^2 + x^2}$

Exercise XXVI.

2. $a - x)\ a^2 + x^2\ (a + x + \dfrac{2x^2}{a - x}$

$$
\begin{array}{l}
a^2 - ax \\
\hline
\quad ax + x^2 \\
\quad ax - x^2 \\
\hline
\qquad 2x^2
\end{array}
$$

3. $x + y)\ x^2 + 2xy + y^2 + x^3 - y^4\ (x + y + x^2 - xy + y^2 - \dfrac{y^3 + y^4}{x + y}$

$$
\begin{array}{l}
x^2 + xy \\
\hline
\quad xy + y^2 \\
\quad xy + y^2 \\
\hline
\qquad x^3 - y^4 \\
\qquad x^3 + x^2y \\
\hline
\qquad\quad - x^2y - y^4 \\
\qquad\quad - x^2y - xy^2 \\
\hline
\qquad\qquad xy^2 - y^4 \\
\qquad\qquad xy^2 + y^3 \\
\hline
\qquad\qquad\quad - y^3 - y^4
\end{array}
$$

4. $m - p)\ 5m^3 - 5p^3 + 3\ (5m^2 + 5mp + 5p^2 + \dfrac{3}{m - p}$

$$
\begin{array}{l}
5m^3 - 5m^2p \\
\hline
\quad 5m^2p - 5p^3 + 3 \\
\quad 5m^2p - 5mp^2 \\
\hline
\qquad 5mp^2 - 5p^3 + 3 \\
\qquad 5mp^2 - 5p^3 \\
\hline
\qquad\qquad 3
\end{array}
$$

5. $ab - b)\ a^2b - ab - a + 1\ (a - \dfrac{a-1}{b(a-1)} = a - \dfrac{1}{b}$

$$\begin{array}{l} a^2b - ab \\ \hline \quad - a + 1 \end{array}$$

6. $m + b)\ m + ab + 5am\ (1 + 5a - \dfrac{b(4a+1)}{m+b}$

$$\begin{array}{l} m + b \\ \hline \quad 5am + ab - b \\ \quad 5am + 5ab \\ \hline \qquad - 4ab - b \end{array}$$

Exercise XXVII.

6. $\dfrac{2(x+y)\,3x}{2(x^2-y^2)};\ \dfrac{2(4x+y)}{2(x^2-y^2)};\ \dfrac{(x-y)(2x-3y)}{2(x^2-y^2)}$, &c.

8. $\dfrac{a}{1}, \dfrac{4x}{3}, \dfrac{x^2+1}{x^2-1}$, and $\dfrac{3x+2}{3}, = \dfrac{3a(x^2-1)}{3(x^2-1)}, \dfrac{4x(x^2-1)}{3(x^2-1)}$,

$\dfrac{3(x^2+1)}{3(x^2-1)}$, and $\dfrac{(3x+2)(x^2-1)}{3(x^2-1)}$.

9. $\dfrac{6a^2(a-b)}{6a^2(a^2-b^2)}, \dfrac{2a}{6a^3(a^2-b^2)}$, and $\dfrac{a-b}{6a^3(a^2-b^2)}$

Exercise XXVIII.

1. $\dfrac{4am + 3m - 2bc}{2bm}$ 2. $\dfrac{y(x+3)\,x + 2(a-b)}{y^2(x+3)}$

3. $\dfrac{(a-b)^2 - (a+b)^2}{a^2 - b^2} = \dfrac{-4ab}{a^2 - b^2} = \dfrac{4ab}{b^2 - a^2}$

4. $\dfrac{315x - 18x + 35x + 63x^2}{63}$

5. $\dfrac{x^3 + y\,(x+y)^2 - xy\,(x+y)}{(x+y)^3} = \dfrac{x^3 + x^2y + 2xy^2 + y^3 - x^2y - xy^2}{(x+y)^3}$

6. $\dfrac{c\,(a-b) + a\,(b-c) - b\,(a-c)}{abc} = \dfrac{0}{abc} = 0.$

7. $\dfrac{m(m-p)-p(m+p)}{(m+p)(m-p)} = \dfrac{m^2-mp-mp-p^2}{(m+p)(m-p)}$, &c.

8. $\dfrac{3(2a-1)-4(1-5a)-7(2a+1)}{4a^2-1} = \dfrac{12a-14}{4a^2-1} = \dfrac{14-12a}{1-4a^2}$

9. Multiplying both num. and den. of 1st fract. by -1 in order to change the signs of the den. we get

$$\frac{x(x-16)+(x+2)(2x+3)-(2-3x)(2-x)}{4-x^2}$$

$$=\frac{x^2-16x+(2x^2+7x+6)-(4-8x+3x^2)}{4-x^2} = \&c.$$

10. $\dfrac{x+y}{a}+\dfrac{x+y}{b}-\dfrac{x+y}{a}+\dfrac{x-y}{b}=\dfrac{x+y}{b}-\dfrac{x-y}{b}=\dfrac{x+y-x+y}{b}=\&c.$

11. $\dfrac{(m+p)(m-p)+(p+x)(p-x)+(m+x)(x-m)}{(p-x)(x-m)(m-p)}$

$$=\frac{(m^2-p^2)+(p^2-x^2)+(x^2-m^2)}{(p-x)(x-m)(m-p)}=\frac{0}{(p-x)(x-m)(m-p)}=0.$$

12. $\dfrac{(a-b)(b+c)+(b-c)(a+b)}{(a+b)(b+c)} - \dfrac{2ab-2ac}{ab+bc+ac+bc-bc+b^2}$

$$=\frac{2ab-2bc-2ab+2ac}{ab+ac+bc+b^2} = \&c.$$

13. $\dfrac{1+x-(1-x)}{1-x^2}+\dfrac{3(1+2x)-3(1-2x)}{1-4x^2}=\dfrac{2x}{1-x^2}+\dfrac{12x}{1-4x^2}$

$$=\frac{2x-8x^3+12x-12x^3}{(1-x^2)(1-4x^2)}=\frac{14x-20x^3}{1-5x^2+4x^4}$$

14. Multiplying both terms of each of the last two frac. by -1 we get

$$\frac{m}{a(a-b)(a-c)}-\frac{m}{b(a-b)(b-c)}-\frac{m}{c(a-c)(c-b)}$$

$$=\frac{bcm(b-c)(c-b)-acm(a-c)(c-b)-abm(b-c)(a-b)}{abc(a-b)(a-c)(b-c)(c-b)}$$

$$=\frac{m(2b^2c^2-b^3c-bc^3-a^2c^2+2a^2bc+ac^3-abc^2-a^2b^2+ab^3-ab^2c)}{abc(2b^2c^2-b^3c-bc^3-a^2c^2+2a^2bc+ac^3-abc^2-a^2b^2+ab^3-ab^2c)}=\frac{m}{abc}$$

Otherwise thus

Multiplying both terms of 2nd fraction *once* by -1, and of 3rd fraction *twice* by -1, we get

$$\frac{m}{a(a-b)(a-c)} - \frac{m}{b(a-b)(b-c)} + \frac{m}{c(a-c)(b-c)}$$

whence we have *l. c. m.* of the den. $= abc(a-b)(b-c)(a-c)$

$$\therefore \text{ the given fractions } = \frac{bcm(b-c) - acm(a-c) + abm(a-b)}{abc(a-b)(a-c)(b-c)}$$

$$= \frac{b^2cm - bc^2m - a^2cm + ac^2m + a^2bm - ab^2m}{abc(a-b)(a-c)(b-c)}$$

$$= \frac{ac^2m - bc^2m - a^2cm + b^2cm + a^2bm - ab^2m}{abc(a-b)(a-c)(b-c)}$$

$$= \frac{c^2m(a-b) - cm(a^2-b^2) + abm(a-b)}{abc(a-b)(a-c)(b-c)} = \frac{c^2m - cm(a+b) + abm}{abc(a-c)(b-c)}$$

$$= \frac{m(c^2 - ac - bc + ab)}{abc(a-c)(b-c)} = \frac{m\{(ab-bc) - (ac-c^2)\}}{abc(a-c)(b-c)}$$

$$= \frac{m\{b(a-c) - c(a-c)\}}{abc(a-c)(b-c)} = \frac{m(b-c)(a-c)}{abc(a-c)(b-c)} = \frac{m}{abc}$$

Exercise XXIX.

1. $\dfrac{2x \times 3x}{5 \times 2a} = \dfrac{3x^2}{5a}$

2. $\dfrac{2m \times x^2 \times y^2}{xy \times my \times x} = 2$

3. $\dfrac{2(a+b)}{xy} \times \dfrac{x(a-b)}{3(a+b)} = \dfrac{2a-2b}{3y}$

4. $\dfrac{3a}{1} \times \dfrac{x+1}{2a} \times \dfrac{x-1}{a+b} = \dfrac{3(x+1)(x-1)}{2(a+b)} = \dfrac{3x^2-3}{2a+2b}$

5. $\dfrac{(a-x)(a+x)}{a+b} \times \dfrac{(a+b)(a-b)}{a+x} \times \dfrac{a}{x(a-x)} = \dfrac{a(a-b)}{x}$

6. $\dfrac{a^2-m^2}{my} \times -\dfrac{a^2+m^2}{a-m} = -\dfrac{(a+m)(a^2+m^2)}{my} = -\dfrac{a^3+ma^2+m^2a+m^3}{my}$

7. $\dfrac{(a-x)(a+x)}{3ax} \times \dfrac{4ax^2}{a+x} = \dfrac{4x(a-x)}{3} = \dfrac{4ax-4x^2}{3}$

8. $\dfrac{(x-7)(x-6)}{x(x-5)} \times \dfrac{(x-5)(x-4)}{x(x-6)} = \dfrac{(x-7)(x-4)}{x^2} = \dfrac{x^2-11x+28}{x^2}$

9. $\dfrac{abcdm}{bcdf^2y^{15}} = \dfrac{am}{f^2y^{15}}$

10. $\dfrac{(a-2)(a+2)}{(a^2-1)} \times \dfrac{a^2-1}{2a} \times \dfrac{a-2}{a+2} = \dfrac{(a-2)(a-2)}{2a} = \dfrac{(a-2)^2}{2a}$

11. $\dfrac{(x-a)(x+a)}{x(x+b)-a(x+b)} \times \dfrac{x(x+b)+c(x+b)}{x(x+c)+d(x+c)}$

$= \dfrac{(x-a)(x+a)}{(x+b)(x-a)} \times \dfrac{(x+b)(x+c)}{(x+c)(x+d)} = \dfrac{x+a}{x+d}$

12. $\dfrac{(x+4)(x-3)}{(x-8)(x-5)} \times \dfrac{(x-5)(x+7)}{(x+4)(x-11)} = \dfrac{(x-3)(x+7)}{(x-8)(x-11)} = \dfrac{x^2+4x-21}{x^2-19x+88}$

13. $\dfrac{1-a+a^2}{1} \times \dfrac{a^2+a+1}{a^2} = \dfrac{\{(a^2+1)-a\}\{(a^2+1)+a\}}{a^2}$

$= \dfrac{(a^2+1)^2-a^2}{a^2} = \dfrac{a^4+a^2+1}{a^2}$

14. $\dfrac{(2a+4m)(2a-4m)}{a-2m} \times \dfrac{5a}{5(2a+4m)(2a+4m)} \times \dfrac{a+2m}{a}$

$= \dfrac{2(a-2m)(a+2m)}{2(a-2m)(a+2m)} = \dfrac{1}{1} = 1$

Exercise XXX.

1. $\dfrac{1}{x} \div \dfrac{x}{y} = \dfrac{1}{x} \times \dfrac{y}{x} = \dfrac{y}{x^2}$

2. $\dfrac{a+x}{a} \div \dfrac{a-x}{a} = \dfrac{a+x}{a} \times \dfrac{a}{a-x} = \dfrac{a+x}{a-x}$

3. $\dfrac{a+b}{a-b} \times \dfrac{(a-b)^2}{(a+b)^2} = \dfrac{a-b}{a+b}$

4. $\dfrac{(a^2+x^2)(a+x)(a-x)}{y+2} \times \dfrac{(y+2)(y-2)}{a-x} \times \dfrac{3a}{a^2+x^2} = 3a(a+x)(y-2)$

5. $\dfrac{x-3}{x-9} \times \dfrac{(x-9)(x-8)}{(x-8)(x-7)} = \dfrac{x-3}{x-7}$

6. $\dfrac{a^2 + b^2}{a^2 - b^2} \div \dfrac{a^2 + b^2}{a^2 - b^2} = \dfrac{a^2 + b^2}{a^2 - b^2} \times \dfrac{a^2 - b^2}{a^2 + b^2} = \dfrac{1}{1} = 1$

7. $\dfrac{(a^3 - x^3)(a^3 + x^3)}{(a - x)^2} \times \dfrac{1}{a + x} \times \dfrac{a - x}{a^2 + ax + x^2} \times \dfrac{1}{a^2 - ax + x^2}$

$= \dfrac{(a - x)(a^2 + ax + x^2)(a + x)(a^2 - ax + x^2)}{(a - x)(a - x)} \times \dfrac{1}{a + x} \times \dfrac{a - x}{a^2 + ax + x^2}$

$\times \dfrac{1}{a^2 - ax + x^2} = \dfrac{1}{1} = 1$

8. $\dfrac{3(a^2 - 1)}{2(a + b)} \times \dfrac{2a(a + b)}{x^2 - 1} = \dfrac{3a(a^2 - 1)}{x^2 - 1} = \dfrac{3a^3 - 3a}{x^2 - 1}$

9. $\dfrac{(xy + y^2) + y^2 + x\ (x + y)}{xy + y^2} \div \dfrac{(2xy + 2y^2) + x\ (x + y) - xy}{xy + y^2}$

$= \dfrac{2y^2 + 2xy + x^2}{xy + y^2} \times \dfrac{xy + y^2}{2y^2 + 2xy + x^2} = 1$

10. $\dfrac{4a^2b^2}{a^4 - b^4} \div \dfrac{4ab}{a^2 - b^2} = \dfrac{4a^2b^2}{(a^2 - b^2)(a^2 + b^2)} \times \dfrac{a^2 - b^2}{4ab} = \dfrac{ab}{a^2 + b^2}$

Exercise XXXI.

1. $\dfrac{\frac{a - b}{3}}{\frac{10a + 9b}{15}} = \dfrac{5(a - b)}{10a + 9b}$

2. $\dfrac{\frac{7a - 2x}{7}}{\frac{3}{1}} = \dfrac{7a - 2x}{21}$

3. $\dfrac{\frac{x}{1}}{\frac{a + 2x}{a}} = \dfrac{ax}{a + 2x}$

4. $\dfrac{\frac{21 - 12x}{20}}{\frac{3x - 1}{6}} = \dfrac{3(21 - 12x)}{10(3x - 1)}$

5. $\dfrac{\frac{15 - 6x + 6a}{10}}{\frac{10a + 10x - 6}{15}} = \dfrac{3\ (15 - 6x + 6a)}{2(10a + 10x - 6)}$

6. $\dfrac{\frac{8a}{1 - 4a^2}}{\frac{2 + 8a^2}{1 - 4a^2}} = \dfrac{4a}{1 + 4a^2}$

7. $\dfrac{\frac{-2a}{1 - a^2}}{\frac{2}{1 - a^2}} = -a$

8. $$\frac{\dfrac{a^2+b^2-ab}{b}}{\dfrac{a-b}{ab}} \times \frac{a^2-b^2}{a^3+b^3} = \frac{a(a^2-ab+b^2)}{a-b} \times \frac{(a-b)(a+b)}{(a+b)(a^2-ab+b^2)}$$

9. $$\frac{\dfrac{x^2y^2-1-x^2y^2}{xy}}{1-\dfrac{1}{1-\dfrac{\frac{1}{1}}{\dfrac{xy-1}{xy}}}} = \frac{\dfrac{-1}{xy}}{1-\dfrac{1}{1-\dfrac{xy}{xy-1}}} = \frac{\dfrac{-1}{xy}}{1-\dfrac{\frac{1}{1}}{\dfrac{xy-1-xy}{xy-1}}}$$

$$= \frac{\dfrac{-1}{xy}}{1-\dfrac{xy-1}{-1}} = \frac{\dfrac{-1}{xy}}{1-\dfrac{1-xy}{1}} = \frac{\dfrac{-1}{xy}}{\dfrac{1-1+xy}{1}} = \frac{\dfrac{-1}{xy}}{\dfrac{xy}{1}} = -\frac{1}{x^2y^2}$$

10. $$\frac{\dfrac{a}{b+\dfrac{\frac{c}{1}}{\dfrac{df+e}{f}}}}{\dfrac{adf-ac}{bdf+be+cf}} = \frac{\dfrac{a}{b+\dfrac{cf}{df+e}}}{\dfrac{adf-ac}{bdf+be+cf}} = \frac{\dfrac{\frac{a}{1}}{\dfrac{bdf+be+cf}{df+e}}}{\dfrac{adf-ac}{bdf+be+cf}} = \frac{\dfrac{adf+ae}{bdf+be+cf}}{\dfrac{adf-ac}{bdf+be+cf}}$$

$$= \frac{adf+ae}{adf-ac} = \frac{a(df+e)}{a(df-c)} = \text{\&c.}$$

11. $$\frac{\dfrac{2+8m^2}{4m-8m^2}}{\dfrac{4m+8m^2}{-8m}} = \frac{\dfrac{1+4m^2}{m-2m^2}}{\dfrac{1+2m}{-1}} = \frac{-1-4m^2}{m(1-4m^2)} = \frac{1+4m^2}{m(4m^2-1)}$$

Exercise XXXII.

1. $12x + 4x = 84 - 3x$, or $19x = 84$, or $x = 4\frac{8}{19}$

2. $10x - x = 5x + 20$, or $4x = 20$, or $x = 5$

3. $168x - 28x + 12x = 63x - 231 + 84x + 756$, or $5x = 525$, or $x = 105$

4. $30x - 105 + 9x - 3 = 5x + 40 - 30x$, or $64x = 148$, or $x = 2\frac{5}{16}$

5. $56 - 4x + 20 = 84 - 7x + 49$, or $3x = 57$, or $x = 19$

6. $56x - 8x = 21x + 7 + 14x + 84$, or $13x = 91$, or $x = 7$

7. $8x - 65 = 35 + 2x$, or $6x = 100$, or $x = 16\frac{2}{3}$

8. $15x + 45 - 12x - 48 - 960 = -20x - 20$, or $23x = 943$, or $x = 41$

9. $80x - 8x - 76 = 300 - 35x - 55$, or $107x = 321$, or $x = 3$

10. $112x + 480 = 3024 - 39x + 84$, or $151x = 2628$, or $x = 17\frac{61}{151}$

11. $208x - 442 + 308x + 374 = 858x - 4433 + 143x$, or $-485x = -4365$, or $x = 9$

12. $4x + 4 - 3x = 6 + 14 - 3x$, or $4x = 16$, or $x = 4$

13. $360x - 160x + 200 + 48x = 2040 + 60 - 180x + 45x + 15$, or $383x = 1915$, or $x = 5$

14. Multiplying by 12 we get $x + \frac{40x - 60}{7} - \frac{34x - 108}{5} = 12 - \frac{39x + 12}{4} - \frac{36 - 23x}{2}$;

This $\times$ 4 and reduced gives $\frac{160x - 240}{7} - \frac{136x - 432}{5} = 3x - 36$, or $800x - 1200 - 952x + 3024 = 105x - 1260$, or $257x = 3084$, or $x = 12$.

15. $60x + 30x + 15x - 36x + 252 = 120x - 156$, or $51x = 408$, or $x = 8$

16. $336 - 10x + 10 - 776 + 56x = 16x - 3x + 11 - 144$, or $33x = 297$, or $x = 9$

17. $30x + 20x + 15x + 12x + 10x = 60x + 25x + 240$, or $2x = 240$, or $x = 120$

18. $12x - 20 + x + 60 = 9x$, or $4x = -40$, or $x = -10$

19. $36 + 20x - 20x = 86 - \frac{125x + 500}{9x - 16}$, or $\frac{125x + 500}{9x - 16} = 50$, or $125x + 500 = 450x - 800$, or $325x = 1300$, or $x = 4$

20. $331 - 30x - \dfrac{120 + 70x}{9} + 9 + 5x = \dfrac{15x - 65}{8} - \dfrac{55x - 85}{4}$ (I)

= the given equat. × 10

$2720 - 200x - \dfrac{960 + 560x}{9} = 15x - 65 - 110x + 170$ (II) = (I) reduced and × 8

$945x + 960 + 560x = 23535$ (III) = II reduced and × 9

$$1505x = 22575, \text{ or } x = 15$$

21. $9x + 20 = \dfrac{144x - 432}{5x - 4} + 9x$ (I) = given equat. × 36

$100x - 80 = 144x - 432$ (II) = I reduced and × $(5x - 4)$

$$44x = 352, \text{ or } x = 8$$

22. $30x + 20x + 60 - 15x + 60 = 12x + 60 + 1900$, or $23x = 1840$, or $x = 80$

23. $90x - 35x - 70 = 75 + 20x + 10 - 51 + 9x$, or $26x = 104$, or $x = 4$

24. $15x + 10x^2 - 10x^2 + 18 = 27 + 18x - \dfrac{12x^2 + 36x + 27}{3 + 4x}$ (I) = given equat. × $(3 + 2x)$

$3x + 9 = \dfrac{12x^2 + 36x + 27}{3 + 4x}$ (II) = (I) transp. and collected

$9x + 27 + 12x^2 + 36x = 12x^2 + 36x + 27$ (III) = (II) × $(3 + 4x)$

$$\therefore 9x = 0, \text{ or } x = 0$$

25. $6x + 12 - \dfrac{21x - 39}{1 + 2x} = 6x + 7$ (I) = given equat. × 9

$5 + 10x = 21x - 39$ (II) = (I) red. and × $(1 + 2x)$; $11x = 44$ $\therefore x = 4$

26. $ax = c - b$, or $x = \dfrac{c - b}{a}$

27. $9ax - 3b^2 = 3bc - 2ax$, or $11ax = 3bc + 3b^2$, or $x = \dfrac{3b^2 + 3bc}{11a}$

28. $8bx - 6x = a - b^2 + 3ax$, or $(8b - 6 - 3a)x = a - b^2$, or $x = \dfrac{a - b^2}{8b - 6 - 3a}$

29. $4a^3bx - 6a^2 + 2ax = 2abx - abx + b^2x$ (I) = given equa. × $2ab$

$(4a^3b + 2a - ab - b^2)x = 6a^2$ (II) = (I) transp. and bracketed

$$\therefore\ x = \frac{6a^2}{4a^3b + 2a - ab - b^2}$$

30. $15abc - 10cx - 5ac = 20ab - 15bx - abcx + b^2c$ (I) = given equat. × $5bc$

$(15b + abc - 10c)x = 20ab + b^2c + 5ac - 15abc$ (II) = (I) transposed and bracketed

$$\therefore\ x = \frac{20ab + b^2c + 5ac - 15abc}{15b + abc - 10c}$$

31. $bdx + adx + bcx = bdf$, or $(bd + ad + bc)x = bdf$, &c.

32. $abx + 4a^2 - 4a^2 + 12bx - 4abx = 4a^2b^2 - 10a^2 + 12bx + 4a^2x$, by multiplying the given equation by $4a$; and this reduced and ÷ by a gives $3bx + 4ax = 10a - 4ab^2$, or $(3b + 4a)x = 10a - 4ab^2$

$$\therefore\ x = \frac{10a - 4ab^2}{3b + 4a}$$

33. $abx - a^2x - b^2c + abc = b^2x$, or $(ab - a^2 - b^2)x = b^2c - abc$,

$$\text{or } x = \frac{bc(b - a)}{ab - a^2 - b^2}$$

34. $11a^2 - 3ax - 11ab + 3bx - (6a^2 + 6ab - 5ax - 5bx)$
$= (a + b)^2 + 2x$ (I) = given equat. × $(a^2 - b^2)$

$2ax + 8bx - 2x = b^2 + 19ab - 4a^2$ (II) = (I) reduced and transp.

$(2a + 8b - 2)x = b^2 + 19ab - 4a^2$, or $x = \dfrac{b^2 + 19ab - 4a^2}{2a + 8b - 2}$

35. $a^2 + 2ax + x^2 - 4abx = x^2$, or $(4b - 2)x = a$, or $x = \dfrac{a}{4b - 2}$

36. $\dfrac{3abc}{a+b} - \dfrac{bx}{a}\left(1 - \dfrac{2ab + b^2}{a^2 + 2ab + b^2}\right) + \dfrac{a^2b^2}{(a + b)^3} = 3cx$ (I)

= given equa. with num. and den. of 1st term × 3, and 2nd and 5th terms factored

$\dfrac{3abc}{a+b} - \dfrac{bx}{a}\left\{\dfrac{a^2}{(a + b)^2}\right\} + \dfrac{a^2b^2}{(a + b)^3} = 3cx$ (II) = (I) with 2nd term red

$\frac{3abc}{a+b} - \frac{abx}{(a+b)^2} + \frac{a^2b^2}{(a+b)^3} = 3cx$ (III) = (II) with 2d term further red.

$\frac{ab}{a+b}\left\{3c + \frac{ab}{(a+b)^2}\right\} = x\left\{3c + \frac{ab}{(a+b)^2}\right\}$ (IV) = (III) with 1st and 3rd, and 2nd and 4th terms factored

$$\therefore x = \frac{ab}{a+b} \text{ (V)} = \text{(IV)} \div \left\{3c + \frac{ab}{(a+b)^2}\right\}$$

37. $3000 + 1720x - 2210x = 203x$ (I) = given equa. × 1000

$$693x = 3000, \text{ or } x = 4\tfrac{76}{231}$$

38. $\frac{3x}{9} + 6x - ax = 3a - \frac{23x}{99}$, or $33x + 594x - 99ax = 297a - 23x$; or $650x - 99ax = 297a$, or $(650 - 99a)x = 297a$, or $x = \frac{297a}{650 - 99a}$

39. $42(x - \frac{1}{3}) + 35(1 - x - \frac{2}{5}) - 30\left(x - 1 - \frac{x}{3}\right) = 105x + 30x$, by multiplying the given equation by 105; and removing the brackets from this we get $42x - 14 + 35 - 35x - 14 - 30x + 30 + 10x = 135x$; or $148x = 37$ $\therefore x = \frac{1}{4}$

40. $72ax - 9b - 75b = 180 - 45b - 35c$, or $72ax = 180 + 39b - 35c$

$$\therefore x = \frac{180 + 39b - 35c}{72a}$$

41. $a^2b^2 + a^2x - b^2x - x^2 - 3ab + 3abx = cx - ac + ax - x^2$

$$a^2x - b^2x + 3abx - cx - ax = 3ab - ac - a^2b^2$$

$$(a^2 - b^2 + 3ab - c - a)x = 3ab - ac - a^2b^2$$

$$\therefore x = \frac{3ab - ac - a^2b^2}{a^2 + 3ab - b^2 - c - a}$$

Exercise XXXIII.

1. Let x = greater, then $47 - x$ = the less, and $x - (47 - x) = 13$, or $2x - 47 = 13$

2. Let x = the less, then $x + 21$ = the greater; $\frac{2x + 21}{x} = 3$, or $2x + 21 = 3x$

3. Let x = money; $\frac{2x}{3} + \frac{2x}{7}$ = part paid away; then $x = \frac{2x}{3} + \frac{2x}{7} + \$2{\cdot}50$

4. Let x = the number; then $\frac{x - 21}{8\frac{3}{5}} = 5$

5. Let x = the quotient, then $2x + 3x + 4x = 54$

6. Let x = debts; then $\frac{2x}{5}$ = 1st payment, and $\frac{3x}{5}$ = remainder; $\therefore\ \frac{3}{7}$ of $\frac{3x}{5} = \frac{9x}{35}$ = 2nd payment; then $\frac{2x}{5} + \frac{9x}{35} + 192 = x$

7. Let x = the number of cattle in the drove, then $\frac{x}{3} + \frac{x}{6} + \frac{x}{5} + 9 = x$

8. Let x = the number of sheep in each flock; $x - 19$ is twice as great as $x - 91$, that is $x - 19 = 2x - 182$

9. Let x = the number; then $\frac{x}{4} - \frac{x}{7} = 6$

10. Let x = the number; then $2x - \frac{3}{7}$ of $\frac{x}{2} = 25$, or $2x - \frac{3x}{14} = 25$

11. Let x = the number; then $x + \frac{x}{2} = 39$

12. Let x = the number; then $x - \left(\frac{x}{2} + \frac{x}{3}\right) = 17$, or $x - \frac{x}{2} - \frac{x}{3} = 17$

13. Let x = the number; then $\frac{2x - 15}{2} + 7 = \frac{3x}{4} + 3$

14. Let x = the number; then $\frac{5(x + 11)}{2} = 85$

15. Let x = the number; then $\frac{x}{2} + \frac{2x}{3} + \frac{3x}{4} = \frac{11x}{8} + 21$

16. Let x = price per barrel; then $\frac{36}{x}$ = number of barrels; and $\frac{36}{x} - 5$ = number of barrels sold the second load; $\left(\frac{36}{x} - 5\right)x = 21$, or $36 - 5x = 21$

17. Let x = distance in miles, then $\frac{1}{2}x$ = half distance; $\frac{1}{2}x \div \frac{7}{2} = \frac{x}{7}$ = times in hours A travels; $\frac{1}{2}x \div 4 = \frac{x}{8}$ = times in hours B travels; then $\frac{1}{7}x - \frac{1}{8}x = \frac{28\frac{1}{2}}{60} = \frac{19}{40}$, or $x = 26\frac{3}{5}$

18. Let x = the time in hours, and since the three runs of stones severally require 72, 84 and 90 hours to empty the granary, they will in 1 hour empty respectively $\frac{1}{72}$, $\frac{1}{84}$ and $\frac{1}{90}$ of it, and in x hours they will empty $\frac{x}{72}$, $\frac{x}{84}$ and $\frac{x}{90}$; similarly the teams will respectively fill in x hours $\frac{x}{60}$ and $\frac{x}{78}$; then

$$\frac{x}{72} + \frac{x}{84} + \frac{x}{90} - \frac{x}{60} - \frac{x}{78} = 1$$

19. Let x = date of abolition of slavery in Canada; then $3(x - 1780) + 1620$ = year of massacre of Lachine. Therefore

$$\frac{x + 3(x - 1780) + 1620}{2} + 116 = 1862$$

20. Let x = A's share, then $x - 120$ = B's, and $x - 106$ = C's. Therefore $x + x - 120 + x - 106 = 7400$

21. Let x = price in cents of a music lesson, then $\frac{24x - 300}{32}$ = price of a drawing lessson; therefore $32x = 24\left(\frac{24x - 300}{32}\right) + 1000$

22. Let x = the number of volumes on science; then $3x$ = number on travels, $3x$ = number on biography; $4\frac{1}{2}x$ = number on history, and $9x$ = number on general literature. Therefore $x + 3x + 3x + \frac{9}{2}x + 9x = 1435$; whence $x = 70$

23. Let x = length of Niagara river, wherefore $4x - 6$ = length of Rideau canal; then $2(5x - 6) - 100 = 230$

24. Let x = days required to finish the work. Then since A does $\frac{1}{12}$, B, $\frac{1}{15}$, and C, $\frac{1}{18}$ of the work in 1 day, A and B working 1 day, and B and C working 2 days will finish $\frac{1}{12} + \frac{3}{15} + \frac{2}{18}$ $= \frac{71}{180}$ of it, and the part remaining to be done $= \frac{109}{180}$; in x days A does $\frac{x}{12}$*ths.*; B, $\frac{x}{15}$*ths.* and C, $\frac{x}{18}$*ths.* of the work, therefore $\frac{x}{12} + \frac{x}{15} + \frac{x}{18} = \frac{109}{180}$, or $15x + 12x + 10x = 109$

25. Let x = greater part; then $n - x$ the less; and $x - (n - x)$ $= a - c$

26. (I) Let x = minute divisions the hour hand passes over; then since the minute hand travels 12 times as fast as the hour hand it will pass over $12x$; but the minute hand also passes completely round the circle (60 minutes), and then in addition over the x minutes. Therefore $60 + x$ is also equal to the number of minute divisions passed over by the minute hand; then $12x = 60 + x$, or $11x = 60$, or $x = 5\frac{5}{11}$; hence the hour $= 5\frac{5}{11}' \times 12 = 1$ h. $5\frac{5}{11}$ m.

(II) To be opposite the hands must be 30 minutes apart; then letting x = space in minutes passed over by hour hand, and remembering that the minute hand travels 12 times as fast, and also goes over $30 + x$ minutes, we have $12x = 30 + x$, or $11x = 30$, or $x = 2\frac{8}{11}$ and $2\frac{8}{11} \times 12 = 32\frac{8}{11}'$ past 12

(III) By similar reasoning to the above $12x = 15 + x$, or $11x = 15$, or $x = 1\frac{4}{11}$, and $1\frac{4}{11}$ m. $\times 12 = 16\frac{4}{11}$ m.

27. Let x = price in dollars of first field; then $x + 90 - 25$ = price of second field, wherefore $(x + 90 - 25) + 90 = 2x$

28. Let x = days required by A and C to finish the remainder; then $\frac{1}{20} - (\frac{1}{50} + \frac{1}{65}) = \frac{19}{1300}$ = part C does in 1 day, ∴ in 11 days C does $\frac{209}{1300}$, and B and C together in 5 days do $\frac{19}{260} + \frac{1}{13} = \frac{39}{260}$

$= \frac{3}{25}$. Hence part remaining to be done $= 1 - (\frac{209}{1300} + \frac{3}{20})$ $= \frac{1300}{1300} - \frac{404}{1300} = \frac{224}{325}$; then in x days A and C will do $\frac{x}{50} + \frac{19x}{1300}$ $\therefore \frac{x}{50} + \frac{19x}{1300} = \frac{224}{325}$, or $26x + 19x = 896$, or $x = 19\frac{41}{45}$ days.

29. Let $x = C$'s share in cents, then $\frac{3x}{7} - 2540 = D$'s share; $\frac{2}{5}(\frac{x}{30} - 2540) + 4000 = \frac{4x}{7} + 2984 = B$'s share, and $(x + \frac{3x}{7} - 2540 + \frac{4x}{7} + 2984) = 2x + 444 = A$'s. Then $x + \frac{3x}{7} - 2540 + \frac{4x}{7} + 2984 + 2x + 444 = 718900$, or $4x + 888 = 718900$, or $4x = 718012$ $\therefore x = \$1795 \cdot 03 = C$'s share; $\frac{3x}{7} - 2540 = \frac{538509}{7} - 2540 = \$743 \cdot 89\frac{6}{7}$, &c.

30. Let x = the number of days required; then since 4 men can do it in 9 days, 1 man can do $\frac{1}{36}$ of it in 1 day; similarly a woman can do $\frac{1}{70}$, and a child $\frac{1}{120}$ of it in 1 day. Hence $\frac{x}{36} + \frac{3x}{70} + \frac{x}{30} = 1$, or $35x + 54x + 42x = 1260$, or $131x = 1260$

31. Let x = right hand digit, then $14 - x$ = the left hand digit and $10(14 - x) + x$ = the number. Hence $\frac{3}{17}(140 - 9x) = \frac{3}{2}x$

32. Let x = value of the property; then $8600 - x$ = gain had the note been good, and $x - (8600 - 640) = x - 7960$ = loss when note proved worthless. Hence $x - 7960 = \frac{2}{5}(8600 - x)$

33. Let x = weight of head; then $x + 9$ = weight of body. Hence $x = 9 + \frac{1}{2}(x + 9)$, or $2x = x + 27$, or $x = 27$ = weight of head; and body $= x + 9 = 27 + 9 = 36$. Hence fish weighs $9 + 27 + 36 = 72$ lbs.

34. Let x = his capital; $x + \frac{1}{3}x - 1000$ = capital at end of 1st year; $\frac{4}{3}x - 1000 + \frac{1}{3}(\frac{4}{3}x - 1000) - 1000$ = capital at end of 2nd year $= \frac{16x - 21000}{9}$; $\frac{16x - 21000}{9} + \frac{1}{3}\left(\frac{16x - 21000}{9}\right) - 1000$

$= \dfrac{64x - 84000}{27} - 1000 =$ capital at end of 3rd year. Hence

$$\frac{64x - 84000}{27} - 1000 = 2x$$

35. Let x = the distance in feet, then $\dfrac{x}{a}$ = number of revolutions of the fore-wheel, and $\dfrac{x}{b}$ = revolutions of the hind-wheel. Hence $\dfrac{x}{a} = \dfrac{x}{b} + n \therefore bx = ax + abn$, whence $bx - ax = abn$ $\therefore$

$$x = \frac{abn}{b-a}$$

36. Let x = number of minute divisions the hour hand passes over before the minute hand overtakes it; then the minute hand must pass from XII to XII, i.e. 60 minutes plus x minutes in order to overtake the hour hand, that is while the hour hand passes over x minute divisions the minute hand passes over $60 + x$ minute divisions, but the minute hand moves through *twelve times* the space the hour hand travels in a given time. Hence $12x$ = the space travelled over by the minute hand, while the hour hand goes over x minutes. Hence $12x = 60 + x \therefore 11x = 60$ and consequently $x = 5\frac{5}{11}$; that is the hands will be together for the first time after XII when the *hour* hand has passed over $5\frac{5}{11}$ of the minute divisions, i. e. in $5\frac{5}{11} \times 12 = 1$ h. $5\frac{5}{11}$ m., and similarly they will be together again 1 h. $5\frac{5}{11}$ m. afterwards, and so on. Hence they will be together at 1 h. $5\frac{5}{11}$ m., 2 h. $10\frac{10}{11}$ m., 3 h. $16\frac{4}{11}$ m., 4 h. $21\frac{9}{11}$ m., &c., and they will be together as often as 1 h. $5\frac{5}{11}$ m. is contained times in 12 h., i.e. 11 times.

37. Let x = the greater part, then $96 - x$ = the less. Hence $\dfrac{x}{7} + 3(96 - x) = 30$; clearing of fractions we have $x + 2016 - 21x = 210$, whence $x = 90\frac{3}{10}$ = the greater, and $96 - x = 96 - 90\frac{3}{10} = 5\frac{7}{10}$ = the less.

38. Let $x = B$'s share, then $\frac{3x}{2} = A$'s share, $\frac{3}{2}$ of $\frac{3x}{2} = \frac{9x}{4} = C$'s share, consequently $x + \frac{3x}{2} + \frac{9x}{4} = 2560$, whence by clearing of fractions $4x + 6x + 9x = 10240$; that is $19x = 10240$, whence $x = \$538{\cdot}94\frac{14}{19} = B$'s share, $\therefore A$'s share $= \frac{3}{2}$ of B's $= \$808{\cdot}42\frac{2}{19}$, and C's share $= \frac{3}{2}$ of A's $= \$1212{\cdot}63\frac{3}{19}$

39. Let x = rate down $\therefore 28x$ = distance, and $x - 5$ = rate up the river, and $x - 3$ = rate up the lake; length of river $= \frac{3}{7}$ of $28x = 12x$, $\therefore$ length of lake $= 16x$

Then $\frac{12x}{x-5} + \frac{16x}{x-3} = \frac{19}{21}\left(\frac{28x}{x-5}\right) \therefore \frac{3}{x-5} + \frac{4}{x-3} = \frac{19}{3(x-5)}$

$\frac{4}{(x-3)} = \frac{10}{3(x-5)} \therefore 12x - 60 = 10x - 30$, $x = 15$, $x - 5 = 10$, $x - 3 = 12$, and $28x = 420$

40. Let x = the whole property, then $\$1800 + \frac{1}{6}(x - 1800) = \$1800 + \frac{x}{6} - \$300 = \frac{x}{6} + \$1500$ = share of the eldest; also $x - \left(\frac{x}{6} + \$1500\right) = \frac{5x}{6} - \$1500$ = part remaining, and $\$3600 + \frac{1}{6}\left(\frac{5x}{6} - \$1500 - \$3600\right) = \$3600 + \frac{5x}{36} - \$850 = \frac{5x}{36} + \$2750$ = share of the second, but these shares are equal. Therefore $\frac{x}{6} + \$1500 = \frac{5x}{36} + \$2750$, whence $x = \$45000$ and $\frac{x}{6} + \$1500 = \$9000$ = share of each; also $\$45000 \div \$9000 = 5$ = number of children.

41. Let x = the left hand digit, then $x + 7$ = the right hand digit; also $10x + x + 7$ = the number, and $x + x + 7 = 2x + 7$ = sum of the digits. Then $\frac{10x + x + 7}{2x + 7} = 2 + \frac{7}{2x + 7}$ whence $11x + 7 = 4x + 14 + 7$, and $\therefore x = 2$, $x + 7 = 9$, consequently the number is 29.

42. Let x = B's share, $x - 20$ = C's, and $\frac{2}{5}(2x - 20) + 80$ = A's. Then $x + x - 20 + \frac{2(2x - 20)}{5} + 80 = 2100$; whence $10x - 100 + 4x - 40 + 400 = 10500$ $\therefore$ $14x = 10240$, and $x = \$731 \cdot 42\frac{6}{7}$ = B's share; also $\$731 \cdot 42\frac{6}{7} - \$20 = \$711 \cdot 42\frac{6}{7}$ = C's share, and $\frac{2}{5}(\$731 \cdot 42\frac{6}{7} + \$711 \cdot 42\frac{6}{7}) + \$80 = \$657 \cdot 14\frac{2}{7}$ = A's share.

43. Let x = the number of rows, then $x^2 + 75$ = number of trees also $x + 6$ rows each containing $x - 5$ trees $= x^2 + x - 30 + 5$ = the number of trees. Then $x^2 + x - 30 + 5 = x^2 + 75$; whence $x = 100$ $\therefore$ $x^2 + 75 = 10000 + 75 = 10075$ = number of trees.

44. Let x = one part, then $a - x$ = the other; and $x = \frac{n}{m}(a - x)$ $\therefore$ $mx = na - nx$, or $mx + nx = na$ $\therefore$ $x = \frac{na}{m + n}$; also $a - x = a - \frac{na}{m+n} = \frac{ma + na - na}{m + n} = \frac{ma}{m+n}$

45. Let x and $60 - x$ = the two parts, x being the less; then $x(60 - x) = 3x^2$ $\therefore$ $60 - x = 3x$, and $x = 15$ = the less; whence $60 - x = 45$ = the greater.

46. Let x = the growth in acres of one acre of grass for one week. Then the growth of $3\frac{1}{3}$ acres for 4 weeks $= x \times \frac{10}{3} \times 4 = \frac{40x}{3}$; and the growth of 10 acres for 9 weeks $= x \times 10 \times 9 = 90x$. Therefore the whole quantity of grass eaten in the first case $= \frac{40x}{3} + 3\frac{1}{3} = \frac{40x + 10}{3}$ acres, and the quantity eaten in the second case $= 90x + 10$

Hence in the first case the quantity of grass eaten by one ox $= \frac{40x + 10}{3} \times \frac{1}{4} \times \frac{1}{12} = \frac{20x + 5}{72}$, and in the second case the quantity of grass eaten by one ox $= (90x + 10) \times \frac{1}{9} \times \frac{1}{21} = \frac{90x + 10}{189}$

But by the question an ox in the first case eats as much as an ox in the second case

Therefore $\dfrac{20x+5}{72} = \dfrac{90x+10}{189}$; whence $x = \frac{1}{12}$ of an acre

Hence $\dfrac{20x+5}{72} = \dfrac{\frac{20}{12}+5}{72} = \dfrac{80}{12 \times 72} = \frac{5}{54}$ = fractional part of an acre of grass eaten by one ox in one week, $\therefore$ one ox in 18 weeks will eat $\frac{5}{54} \times 18 = \frac{5}{3}$ acres.

Now since each acre increases at the rate of $\frac{1}{12}$ of an acre per week 24 acres will increase 2 acres per week, and in 18 weeks the 24 acres increase by 36 acres, and therefore become equivalent to 60 acres

Then 60 acres $\div \frac{5}{3}$ acres = 36 oxen.

47. Let x = the first, then nx = the second, and mx = the third. Therefore $x + nx + mx = a$, whence $x = \dfrac{a}{1+n+m}$ = first; second $= nx = \dfrac{na}{1+m+n}$; and third $= mx = \dfrac{ma}{1+m+n}$

48. Let x = the first, then $\dfrac{mx}{n}$ = the second, and $\dfrac{px}{q}$ = the third. Therefore $x + \dfrac{mx}{n} + \dfrac{px}{q} = a$, whence $nqx + mqx + npx = nqa$, and

$\therefore\ x = \dfrac{anq}{nq+mq+np}$ = the first part

Second part $= \dfrac{mx}{n} = \dfrac{m}{n} \times \dfrac{anq}{nq+mq+np} = \dfrac{amq}{nq+mq+np}$

Third part $= \dfrac{px}{q} = \dfrac{p}{q} \times \dfrac{anq}{nq+mq+np} = \dfrac{anp}{nq+mq+np}$

49. Let x = the number thrown by the first after the second commences; then $x + 36$ = whole number thrown by the first; $\dfrac{7x}{8}$ = the number thrown by the second. But every 4 charges of the first consume as much powder as every 3 charges of the second, and they are to consume equal amounts of powder, $\therefore \dfrac{x+36}{4} = \dfrac{7x}{24}$; whence $x = 216$ = balls thrown by the first

after the second commences. Therefore $\frac{7}{8}$ of $216 = 189 =$ balls thrown by second.

Exercise XXXIV.

10. $$\left.\begin{aligned} 4x + 6y &= 2a \\ 15x - 6y &= 3b \end{aligned}\right\}$$
$$19x = 2a + 3b$$
$$x = \frac{2a + 3b}{19}$$
$$3y = a - 2x$$
$$3y = a - \frac{4a + 6b}{19}$$
$$3y = \frac{19a - 4a - 6b}{19}$$
$$3y = \frac{15a - 6b}{19}$$
$$y = \frac{5a - 2b}{19}$$

11. $$12x + 4ay = 4m$$
$$12x + 3by = 3n$$
$$4ay - 3by = 4m - 3n$$
$$(4a - 3b)y = 4m - 3n$$
$$y = \frac{4m - 3n}{4a - 3b}$$
$$3x = m - ay = m - \frac{4am - 3an}{4a - 3b}$$
$$3x = \frac{4am - 3bm - 4am + 3an}{4a - 3b}$$
$$3x = \frac{3an - 3bm}{4a - 3b}$$
$$x = \frac{an - bm}{4a - 3b}$$

12. $$2abx - 4aby = 2b^2$$
$$2abx - aby = ac$$
$$3aby = ac - 2b^2$$
$$y = \frac{ac - 2b^2}{3ab}$$
$$ax = b + 2ay = b + \frac{2ac - 4b^2}{3b}$$
$$ax = \frac{3b^2 + 2ac - 4b^2}{3b}$$
$$ax = \frac{2ac - b^2}{3b}$$
$$x = \frac{2ac - b^2}{3ab}$$

13. $$\left.\begin{aligned} x - y &= a \\ x^2 - y^2 &= b \end{aligned}\right\}$$
$$\left.\begin{aligned} (x - y)(x + y) &= b \\ a(x + y) &= b \end{aligned}\right\}$$
$$ax + ay = b$$
$$ax - ay = a^2$$
$$2ax = a^2 + b$$
$$x = \frac{a^2 + b}{2a}$$
$$2ay = b - a^2$$
$$y = \frac{b - a^2}{2a}$$

14. $$\left.\begin{aligned} cx - ay &= acm \\ (m-c)x + (m+c)y &= acm \end{aligned}\right\}$$

$$c(m-c)x - a(m-c)y = acm(m-c)$$
$$c(m-c)x + c(m+c)y = ac^2m$$
$$mcy + c^2y + amy - acy = ac^2m - acm^2 + ac^2m$$

$$y = \frac{acm(2c-m)}{mc + c^2 + am - ac}$$

$$cx = acm + ay = acm + \frac{2a^2c^2m - a^2cm^2}{mc + c^2 + am - ac}$$

$$x = \frac{am(mc + c^2 + am - ac) + 2a^2cm - a^2m^2}{mc + c^2 + am - ac}$$

$$x = \frac{am^2c + amc^2 + a^2m^2 - a^2cm + 2a^2cm - a^2m^2}{mc + c^2 + am - ac}$$

$$x = \frac{am^2c + amc^2 + a^2cm}{mc + c^2 + am - ac}$$

15. $$\frac{bm}{x} + \frac{bn}{y} = ab$$

$$\frac{bm}{x} - \frac{mq}{y} = bm$$

$$\frac{bn}{y} + \frac{mq}{y} = ab - bm$$

$$(ab - bm)y = bn + mq$$

$$y = \frac{bn + mq}{ab - bm}$$

$$\frac{m}{x} = a - \frac{n}{y} = a - \frac{n}{\dfrac{bn + mq}{ab - bm}}$$

$$\frac{m}{x} = a - \frac{abn - bmn}{bn + mq}$$

$$\frac{m}{x} = \frac{abn + amq - abn + bmn}{bn + mq}$$

$$\frac{1}{x} = \frac{aq + bn}{mq + bn}$$

$$x = \frac{mq + bn}{aq + bn}$$

16. $$\left.\begin{aligned} (x-y)(x+y) &= 55 \\ x + y &= 11 \end{aligned}\right\}$$

$$11(x-y) = 55$$
$$x - y = 5$$
$$x + y = 11$$
$$2x = 16$$
$$x = 8$$
$$2y = 6$$
$$y = 3$$

17. $$\left.\begin{aligned} 459x - 463y &= -495 \\ -5x + 19y &= 131 \end{aligned}\right\}$$

$$2295x - 2315y = -2475$$
$$-2295x + 8721y = 60129$$
$$6406y = 57654$$
$$y = 9$$
$$5x = 19y - 131 = 171 - 131$$
$$5x = 40$$
$$x = 8$$

18. $cx - ay = acp$

$$\underline{ax - cy = a^2 + c^2}$$

$$acx - a^2y = a^2cp$$

$$\underline{acx - c^2y = a^2c + c^3}$$

$$c^2y - a^2y = a^2cp - a^2c - c^3$$

$$y = \frac{a^2cp - a^2c - c^3}{c^2 - a^2}$$

$$ax = a^2 + c^2 + cy = a^2 + c^2 + \frac{a^2c^2p - a^2c^2 - c^4}{c^2 - a^2}$$

$$ax = \frac{c^4 - a^4 + a^2c^2p - a^2c^2 - c^4}{c^2 - a^2} \quad \therefore x = \frac{ac^2p - a^3 - ac^2}{c^2 - a^2}$$

(19)

$$\left.\begin{array}{l} 35x - 126 + 28xy + 49y = 28xy + 76y \\ 72x^2 - 90xy - \frac{141}{10}x + 42y - \frac{91}{10} = 72x^2 - 90xy + 930y - 1002x + 2666 \end{array}\right\}$$

$$\left.\begin{array}{r} 35x - 27y = 126 \\ 3293x - 2960y = 8917 \end{array}\right\}$$

$$115255x - 88911y = 414918$$

$$\underline{115255x - 103600y = 312095}$$

$$14689y = 102823 \quad \therefore y = 7$$

$$35x = 126 + 27y = 126 + 189 = 315 \quad \therefore x = 9$$

(20)

$$3(a^2 - b^2)x + 5(a^2 - b^2)y = 8a^2b - 2ab^2$$

$$\underline{3(a^2 - b^2)x + 3(a + b + c)by = 3a^2b + 6ab^2 + \frac{3ab^2c}{a + b}}$$

$$5(a^2 - b^2)y - 3(ab + b^2 + bc)y = 8a^2b - 2ab^2 - 3a^2b - 6ab^2 - \frac{3ab^2c}{a + b}$$

$$(5a^2 - 8b^2 - 3ab - 3bc)y = \frac{5a^3b - 3a^2b^2 - 8ab^3 - 3ab^2c}{a + b}$$

$$(5a^2 - 8b^2 - 3ab - 3bc)y = \frac{(5a^2 - 8b^2 - 3ab - 3bc)ab}{(a + b)}$$

$$y = \frac{ab}{a + b}$$

$$3x = \frac{8a^2b - 2ab^2}{a^2 - b^2} - 5y = \frac{8a^2b - 2ab^2}{a^2 - b^2} - \frac{5ab}{a + b} = \frac{8a^2b - 2ab^2 - 5a^2b + 5ab^2}{a^2 - b^2}$$

$$3x = \frac{3a^2b + 3ab^2}{a^2 - b^2} = \frac{3ab(a + b)}{a^2 - b^2}; \quad 3x = \frac{3ab}{a - b} \quad \therefore x = \frac{ab}{a - b}$$

Exercise XXXV.

(6)

$$\frac{1}{x}+\frac{1}{y}=\frac{5}{6}$$

$$\frac{1}{x}+\frac{1}{z}=\frac{3}{4}$$

$$\frac{1}{y}-\frac{1}{z}=\frac{5}{6}-\frac{3}{4}=\frac{1}{12}$$

$$\frac{1}{y}+\frac{1}{z}=\frac{7}{12}$$

$$\frac{2}{y}=\frac{8}{12};\ 8y=24;\ y=3$$

$$\frac{1}{3}+\frac{1}{z}=\frac{7}{12}$$

$$\frac{1}{z}=\frac{7}{12}-\frac{1}{3}=\frac{1}{4}\ \therefore z=4$$

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$$

$$\frac{1}{x}=\frac{5}{6}-\frac{1}{3}=\frac{1}{2}\ \therefore x=2$$

(7)

$$\left.\begin{aligned} x+y&=xy\\ x+z&=2xz\\ 2y+2z&=3yz\end{aligned}\right\}$$

$$\frac{1}{y}+\frac{1}{x}=1$$

$$\frac{1}{z}+\frac{1}{x}=2$$

$$\frac{1}{z}-\frac{1}{y}=1$$

$$\frac{2}{z}-\frac{2}{y}=2$$

$$\frac{2}{z}+\frac{2}{y}=3$$

$$\frac{4}{z}=5\ \therefore z=\frac{4}{5}$$

$$\frac{4}{y}=1\ \therefore y=4$$

$$\frac{1}{x}+\frac{1}{y}=1$$

$$\frac{1}{x}+\frac{1}{4}=1\ \therefore x=\frac{4}{3}$$

(8)

$$\left.\begin{aligned} x+3y+2z&=b\\ 3x+5y-2z&=m\\ 8x-2y+2z&=2n\end{aligned}\right\}$$

$$\left.\begin{aligned} 4x+8y&=b+m\\ 11x+3y&=m+2n\end{aligned}\right\}$$

$$12x+24y=3b+3m$$

$$88x+24y=8m+16n$$

$$76x=5m+16n-3b$$

$$x=\frac{5m+16n-3b}{76}$$

$$\frac{5m+16n-3b}{19}+8y=b+m$$

(10)

Add all four equations together and then ÷ 3

$$v+x+y+z=23$$

$$v+x+y=13$$

$$z=10$$

$$v+x+z=17$$

$$y=6$$

$$v+y+z=18$$

$$x=5$$

$$x+y+z=21$$

$$v=2$$

$$8y = b + m - \frac{5m + 16n - 3b}{19}$$

$$8y = \frac{22b + 14m - 16n}{19}; \quad y = \frac{11b + 7m - 8n}{76}$$

$$z = n - 4x + y; \quad z = n - \frac{5m + 16n - 3b}{19} + \frac{11b + 7m - 8n}{76}$$

$$z = \frac{76n - 20m - 64n + 12b + 11b + 7m - 8n}{76}$$

$$z = \frac{23b + 4n - 13m}{76}$$

(9)

$$abx + b^2y = bc$$
$$abx + acz = a^2$$
$$b^2y - acz = bc - a^2$$
$$c^2y + acz = bc$$
$$(b^2 + c^2)y = 2bc - a^2; \quad y = \frac{2bc - a^2}{b^2 + c^2}$$

$$az = b - cy = b - \frac{2bc^2 - a^2c}{b^2 + c^2}; \quad az = \frac{b^3 + bc^2 - 2bc^2 + a^2c}{b^2 + c^2}$$

$$z = \frac{b^3 - bc^2 + a^2c}{ab^2 + ac^2}$$

$$ax = c - by = c - \frac{2b^2c - ba^2}{b^2 + c^2}; \quad ax = \frac{cb^2 + c^3 - 2b^2c + a^2b}{b^2 + c^2}$$

$$x = \frac{c^3 - b^2c + a^2b}{ab^2 + ac^2}$$

(11)

$$ax + ay + az = a^2 + ab + ac$$
$$bx + cy + az = a^2 + b^2 + c^2$$
$$(a - b)x + (a - c)y = ab + ac - b^2 - c^2$$
$$bx + by + bz = ab + b^2 + bc$$
$$cx + ay + bz = a^2 + b^2 + c^2$$
$$(b - c)x + (b - a)y = ab + bc - a^2 - c^2$$

$$\left.\begin{aligned}(a - b)x + (a - c)y &= ab + ac - b^2 - c^2\\(b - c)x + (b - a)y &= ab + bc - a^2 - c^2\end{aligned}\right\}$$

$$(2ab - b^2 - a^2)x + (a - c)(b - a)y = 2ab^2 + abc - b^3 - bc^2 - a^2b - a^2c + ac^2$$
$$(ab - ac - bc + c^2)x + (a - c)(b - a)y = a^2b - a^3 - ac^2 - bc^2 + a^2c + c^3$$

$$(b^2+a^2+c^2-ab-ac-bc)x=b^3+c^3-a^3-2ab^2-abc+2a^2b+2a^2c-2ac^2$$

$$x=\frac{b^3+c^3-a^3-2ab^2-2ac^2+2a^2b+2a^2c-abc}{b^2+a^2+c^2-ab-ac-bc}=b+c-a$$

$$(a-c)y=ab+ac-b^2-c^2-(a-b)(b+c-a)$$

$$(a-c)y=a^2-ab-c^2+bc \therefore y=a+c-b$$

$$x+y+z=a+b+c \therefore z=a+b+c-(b+c-a)-(a+c-b)$$

$$z=a+b+c-b-c+a-a-c+b;\quad z=a+b-c$$

(12)

$$\left.\begin{array}{l} ax+a^2y+a^2z=am \\ ax+\ \ y+az\ =n \\ ax+ay+\ \ z\ =p \end{array}\right\}$$

$$\left.\begin{array}{l} a^2y-y+a^2z-az=am-n \\ ay\ -y+\ \ z-az=p\ \ -n \end{array}\right\}$$

$$\left.\begin{array}{l} (a^2-1)y+(a^2-a)z=am-n \\ (a\ -1)y-(a\ -1)z=p\ \ -n \end{array}\right\}$$

$$\left.\begin{array}{l} (a^2-1)y+(a^2-a)z=am-n \\ (a^2-a)y-(a^2-a)z=ap\ -an \end{array}\right\}$$

$$(2a^2-a-1)y=am-n+ap-an \therefore y-\frac{am-n+ap-an}{2a^2-a-1}$$

$$(a-1)z=(a-1)y-p+n$$

$$(a-1)z=\frac{am-n+ap-an}{2a+1}-p+n$$

$$(a-1)z=\frac{am-n+ap-an-2ap+2an-p+n}{2a+1}$$

$$(a-1)z=\frac{am-ap+an-p}{2a+1} \therefore z=\frac{am-ap+an-p}{2a^2-a-1}$$

$$x=m-a(y+z)=m-a\left(\frac{am-n+ap-an+am-ap+an-p}{2a^2-a-1}\right)$$

$$x=m-a\left(\frac{2am-n-p}{2a^2-a-1}\right) \therefore x=m-\frac{2a^2m-an-ap}{2a^2-a-1}$$

$$x=\frac{2a^2m-am-m-2a^2m+an+ap}{2a^4-a-1} \therefore x=\frac{ap-am+an-m}{2a^2-a-1}$$

Exercise XXXVI.

1. $7(x + y) + 4y = 50$, and $2(x - y) + 3x = 16$

2. $x + y = a$, and $bx - cy = 0$

3. Let x = price in cents of hay per ton, and y = price of oats per bushel: $2x + 35y = 4400$, also $\frac{4y}{5}$ = reduced price of oats and $\frac{4x}{3}$ = increased price of hay; then $\frac{8x}{3} + 28y = 5120$. Hence the equations are $2x + 35y = 4400$, and $\frac{8x}{3} + 28y = 5120$

4. Let x = length, and y = breadth; then xy = area. Then $(x + 20)(y + 24) = xy + 4180$, and $(x + 24)(y + 20) = xy + 3860$, which two equations when reduced give $6x + 5y = 925$, and $5x + 6y = 845$

5. $\frac{1}{2}x + \frac{1}{3}y = 11$; $\frac{1}{3}x - 1 = \frac{1}{5}y$

6. $x + y = 144$, and $\frac{4}{7}x - \frac{7}{9}y = 1\frac{1}{3}$

7. $x + y = 48$, and $\frac{x}{4} = \frac{4x}{y}$, or $xy = 16x$, or dividing by x we get $y = 16$

8. $x = \frac{5}{4}(y + z)$; $y = \frac{1}{2}(x + z) + 6$; $z = \frac{1}{3}(x + y) - 3$; whence by reduction we get $4x - 5y - 5z = 0$; $-x + 2y - z = 12$, and $-x - y + 3z = -9$

9. Let x = sulphur, y = saltpetre, and z = charcoal
$x + y + z = 4000$; $y + z - x = 3240$; $x + y - z = 2760$

10. $x + y + z = 72$; $\frac{1}{2}x = \frac{1}{3}y$; $\frac{1}{2}x = \frac{1}{4}z$

11. Let x = space occupied by one shilling, and y = space filled by a ten cent piece; then $16x + 27y = 1$, and $11x + 13y = \frac{9}{16}$, whence $x = \frac{35}{1424}$, and $y = \frac{2}{89}$; wherefore the purse would hold $\frac{1424}{35} = 40\frac{24}{35}$ shillings, or $\frac{89}{2} = 44\frac{1}{2}$ ten cent pieces.

12. Let x = number of lines, and y = number of letters in a line; then xy = number of letters on a page. Then $(x+3)(y+4) = xy + 224$, and $(x-2)(y-3) = xy - 145$; or $4x + 3y = 212$, and $3x + 2y = 151$

13. Let x = left hand digit, and y = right hand one; then the number will be represented by $10x + y$. Whence $\frac{10x+y}{2x+2y-4} = 3$, and $\frac{10x+y}{y-x+5} = 13$; or $4x - 5y = -12$, and $23x - 12y = 65$

14. Let x = number of ten cent pieces, and y = number of twenty-five cent pieces; then $10x + 25y = 8160$, and $\frac{5y}{2} - 6x = 4$; or $5y - 12x = 8$, and $5y + 2x = 1632$

15. Let x = rate before the accident $\therefore x - \frac{x}{a} = \frac{x(a-1)}{a}$ = rate after the accident

Let n = the number of miles from Kingston at which the accident occurred.

$$\therefore \frac{n}{\frac{x(a-1)}{a}} = \frac{n}{x} + b \quad \therefore \frac{an}{x(a-1)} = \frac{n}{x} + b \text{ (I)}$$

And $$\frac{n-c}{\frac{x(a-1)}{a}} = \frac{n-c}{x} + d \quad \therefore \frac{a(n-c)}{x(a-1)} = \frac{n-c}{x} + d \text{ (II)}$$

From (I) $$\frac{an}{x(a-1)} - \frac{n}{x} = \frac{n}{x(a-1)} = b \text{ (III)}$$

From (II) $$\frac{a(n-c)}{x(a-1)} - \frac{n-c}{x} = \frac{n-c}{x(a-1)} = d \text{ (IV)}$$

$\therefore$ From (IV) $\frac{n}{x(a-1)} - \frac{c}{x(a-1)} = d$. But (III) $\frac{n}{x(a-1)} = b$

$$\therefore b - \frac{c}{x(a-1)} = d; \text{ or } bx - \frac{c}{a-1} = dx$$

whence $x(b-d) = \frac{c}{a-1} \quad \therefore x = \frac{c}{(a-1)(b-d)}$

16. Let x = the number of inside passengers, and y = the fare in dollars of each

Then $x + 4$ = number of outside passengers, and $\frac{1}{7}(4y - \frac{1}{2})$ = fare of each

Then $xy + (x + 4) \times \frac{1}{7}(4y - \frac{1}{2}) = 45$; or $22xy + 32y - x = 634$

Also $\frac{1}{2}y$ = fare of inside passengers for half way, and $\frac{1}{14}(4y - \frac{1}{2})$ = fare of outside passengers for do., and the whole fare was increased by $\frac{2}{15}$ of $\$45 = 6$

Then $\frac{1}{2}y + \frac{3}{14}(4y - \frac{1}{2}) = 6$; or $38y = 171$; or $y = \$4\frac{1}{2}$, and this substituted in the first equation for y gives us $22 \times \frac{9}{2}x + 32 \times \frac{9}{2} - x = 634$, or $98x = 490$; whence $x = 5$ = inside passengers

17. Let x and y = the digits; then the number will be $10x + y$

Then $10x + y = 2xy$ (I) and $10x + y = 4x + 4y$; or $6x - 3y = 0$; or $2x - y = 0$ (II)

Adding equations (I) and (II) we have $12x - 2xy = 0$, and dividing this by $2x$, and transposing, we have $y = 6$; whence $x = 3$, and the number $= 36$

18. Let x, y and z be the digits, then the number will be $100x + 10y + z$. Then $y = \frac{1}{2}(x + z)$; $\frac{100x + 10y + z}{x + y + z} = 48$, and $100x + 10y + z - 198 = 100z + 10y + x$. These reduced give the equations $-x + 2y - z = 0$, $52x - 38y - 47z = 0$, and $x - z = 2$, &c.

19. Let x = the oz. of A, and y = oz. of B; then $x + y = p$ (I) since p oz. of A lose b oz. in water, 1 oz. will lose $\frac{b}{p}$, and $\therefore x$ oz. lose $\frac{bx}{p}$; similarly y oz. of B lose $\frac{cy}{p}$ oz. in water $\therefore \frac{bx}{p} + \frac{cy}{p} = a$ (II)

From (II) $bx + cy = ap$ (III) and multiplying (I) by b we get

$bx + by = bp$ (IV); then (III) – (IV) gives us $cy - by = ap - bp$

$$\therefore y = \frac{(a - b)p}{c - b}; \text{ similarly } x = \frac{(c - a)p}{c - b}$$

20. Let their money at starting be represented respectively by v, w, x, y, and z.

Then $v + w + x + y + z = 160$, since each had \$32 at end, their money must be equal to 32×5

	A	B	C	D	E
1st game,	$v - (w + x + y + z)$	$2w$	$2x$	$2y$	$2z$
2nd game,	$2(v - w - x - y - z)$	$2w - (v - w + x + y + z)$	$4x$	$4y$	$4z$
3rd game,	$4(v - w - x - y - z)$	$2(3w - v - x - y - z)$	$4x - (v + w - 3x + y + z)$	$8y$	$8z$
4th game,	$8(v - w - x - y - z)$	$4(3w - v - x - y - z)$	$2(7x - v - w - y - z)$	$8y - (v + w + x - 7y + z)$	$16z$
5th game,	$16(v - w - x - y - z)$	$8(3w - v - x - y - z)$	$4(7x - v - w - y - z)$	$2(15y - v - w - x - z)$	$16z$ – &c.

$v + w + x + y + z = 160$ (I) as above

Consequently $16(v - w - x - y - z) = 32 \therefore v - w - x - y - z = 2$ (II)

Also $8(3w - v - x - y - z) = 32 \therefore 3w - v - x - y - z = 4$ (III)

$4(7x - v - w - y - z) = 32 \therefore 7x - v - w - y - z = 8$ (IV)

And $2(15y - v - w - x - z) = 32 \therefore 15y - v - w - x - z = 16$ (V)

Now (I) + (II) gives $2v = 162 \therefore v = \$81$; Also (I) + (III) gives $4w = 164 \therefore w = \$41$

(I) + (IV) gives $8x = 168 \therefore x = \$21$; (I) + (V) gives $16y = 176 \therefore y = \$11$

And $v + w + x + y + z = 81 + 41 + 21 + 11 + z = 160 \therefore z = 160 - 154 = \$6$

Exercise XXXVIII.

7. $(2a+3)^6 = (2a)^6 + 6(2a)^5 3 + 15(2a)^4 3^2 + 20(2a)^3 3^3 + 15(2a)^2 3^4 + 6(2a)3^5 + 3^6$

8. $(3-2m)^5 = 3^5 - 5 \times 3^4(2m) + 10 \times 3^3(2m)^2 - 10 \times 3^2(2m)^3 + 5 \times 3(2m)^4 - (2m)^5$

9. $(3a-2y)^5 = (3a)^5 - 5(3a)^4(2y) + 10(3a)^3(2y)^2 - 10(3a)^2(2y)^3 + 5(3a)(2y)^4 - (2y)^5$

10. $(2b-5c)^3 = (2b)^3 - 3(2b)^2(5c) + 3(2b)(5c)^2 - (5c)^3$

11. $(3x-4y)^4 = (3x)^4 - 4(3x)^3(4y) + 6(3x)^2(4y)^2 - 4(3x)(4y)^3 + (4y)^4$

12. $(ab+3c)^5 = (ab)^5 + 5(ab)^4(3c) + 10(ab)^3(3c)^2 + 10(ab)^2(3c)^3 + 5(ab)(3c)^4 + (3c)^5$

13. $(2ac - xyz)^3 = (2ac)^3 - 3(2ac)^2(xyz) + 3(2ac)(xyz)^2 - (xyz)^3$

14. $\{(a+b)-c\}^3 = (a+b)^3 - 3(a+b)^2c + 3(a+b)c^2 - c^3 = a^3 + 3a^2b + 3ab^2 + b^3 - 3c(a^2+2ab+b^2) + 3c^2(a+b) - c^3$

15. $\{2a-(b+c)\}^4 = (2a)^4 - 4(2a)^3(b+c) + 6(2a)^2(b+c)^2 - 4(2a)(b+c)^3 + (b+c)^4 = 16a^4 - 32a^3(b+c) + 24a^2(b^2+2bc+c^2) - 8a(b^3+3b^2c+3bc^2+c^3) + b^4 + 4b^3c + 6b^2c^2 + 4bc^3 + c^4$

16. $\{2(a+b)-3c\}^5 = 2^5 \times (a+b)^5 - 5 \times 2^4(a+b)^4(3c) + 10 \times 2^3(a+b)^3(3c)^2 - 10 \times 2^2(a+b)^2(3c)^3 + 5 \times 2(a+b)(3c)^4 - (3c)^5 = 32(a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5) - 240c(a^4+4a^3b+6a^2b^2+4ab^3+b^4) + 720c^2(a^3+3a^2b+3ab^2+b^3) - 1080c^3(a^2+2ab+b^2) + 810c^4(a+b) - 243c^5$

17. $\{(1+x)-x^2\}^4 = (1+x)^4 - 4(1+x)^3(x^2) + 6(1+x)^2(x^2)^2 - 4(1+x)(x^2)^3 + (x^2)^4 = 1+4x+6x^2+4x^3+x^4 - 4x^2(1+3x+3x^2+x^3) + 6x^4(1+2x+x^2) - 4x^6(1+x) + x^8$

18. $\{(a-b)+2c\}^5 = (a-b)^5 + 5(a-b)^4(2c) + 10(a-b)^3(2c)^2 + 10(a-b)^2(2c)^3 + 5(a-b)(2c)^4 + (2c)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5 + 10c(a^4-4a^3b+6a^2b^2-4ab^3+b^4) + 40c^2(a^3-3a^2b+3ab^2-b^3) + 80c^3(a^2-2ab+b^2) + 80c^4(a-b) + 32c^5$

Exercise XXXIX.

(1)

$$\begin{array}{l} 4 + 2x - 12x^2 \\ \qquad \frac{1}{4}x^2 - 3x^3 \\ \qquad\qquad + 9x^4 \\ \hline \end{array}$$

(2)

$$\begin{array}{l} x^2 + 2x^3 - 2x^4 \\ \qquad x^4 - 2x^5 \\ \qquad\qquad + x^6 \\ \hline \end{array}$$

(3)

$$\begin{array}{l} 4x^2 - 12x^3 - 2x^4 \\ \qquad 9x^4 + 3x^5 \\ \qquad\qquad + \frac{1}{4}x^6 \\ \hline \end{array}$$

(4)

$$\begin{array}{l} 1 - a + 4a^2 - 2a^3 \\ \qquad \frac{1}{4}a^2 - 2a^3 + a^4 \\ \qquad\qquad 4a^4 - 4a^5 \\ \qquad\qquad\qquad + a^6 \\ \hline \end{array}$$

(5)

$$\begin{array}{l} 1 + 2x - x^2 - x^3 + 2x^4 \\ \qquad x^2 - x^3 - x^4 + 2x^5 \\ \qquad\qquad \frac{1}{4}x^4 + \frac{1}{2}x^5 - x^6 \\ \qquad\qquad\qquad \frac{1}{4}x^6 - x^7 \\ \qquad\qquad\qquad\qquad + x^8 \\ \hline \end{array}$$

(6)

$$\begin{array}{l} 4a^2 - 4a^2x + 8a^2x^2 \\ \qquad a^2x^2 - 4a^2x^3 \\ \qquad\qquad + 4a^2x^4 \\ \hline \end{array}$$

(7)

$$\begin{array}{l} 1 + 2bx - 2cx^2 \\ \qquad b^2x^2 - 2bcx^3 \\ \qquad\qquad + c^2x^4 \\ \hline \end{array}$$

(8)

$$\begin{array}{l} a^2 - 2abx - 2acx^2 + 2adx^3 \\ \qquad b^2x^2 + 2bcx^3 - 2bdx^4 \\ \qquad\qquad c^2x^4 - 2cdx^5 \\ \qquad\qquad\qquad + d^2x^6 \\ \hline \end{array}$$

(9)

$$\begin{array}{l} 1 - 2a + 2b^2x^2 - 2c^3x^3 + 2d^4x^4 \\ \qquad a^2 - 2ab^2x^2 + 2ac^3x^3 - 2ad^4x^4 \\ \qquad\qquad b^4x^4 - 2b^2c^3x^5 + 2b^2d^4x^6 \\ \qquad\qquad\qquad c^6x^6 - 2c^3d^4x^7 \\ \qquad\qquad\qquad\qquad + d^8x^8 \\ \hline \end{array}$$

(10)

$$\{(a+b)^3\}^2 = (a^3 + 3a^2b + 3ab^2 + b^3)^2 =$$

$$a^6 + 6a^5b + 6a^4b^2 + \quad 2a^3b^3$$

$$9a^4b^2 + 18a^3b^3 + 6a^2b^4$$

$$9a^2b^4 + 6ab^5$$

$$+ b^6$$

(11)

$$\{(a-c)^2\}^4 = (a^2 - 2ac + c^2)^4$$

$$= \{(a^2 - 2ac + c^2)^2\}^2 = (a^4 - 4a^3c + 6a^2c^2 - 4ac^3 + c^4)^2 =$$

$$a^8 - 8a^7c + 12a^6c^2 - \quad 8a^5c^3 + \quad 2a^4c^4$$

$$16a^6c^2 - 48a^5c^3 + 32a^4c^4 - \quad 8a^3c^5$$

$$36a^4c^4 - 48a^3c^5 + 12a^2c^6$$

$$16a^2c^6 - 8ac^7$$

$$+ c^8$$

(12)

$$(a^2x^2 - 4ax + 4)^2 = a^4x^4 - 8a^3x^3 + 8a^2x^2$$

$$16a^2x^2 - 32ax$$

$$+ 16$$

(13)

$$4 - 12x + 16x^2 - \quad 2x^3 + 1\tfrac{1}{3}x^4$$

$$9x^2 - 24x^3 + 3x^4 - 2x^5$$

$$16x^4 - 4x^5 + \tfrac{8}{3}x^6$$

$$\tfrac{1}{4}x^6 - \tfrac{1}{3}x^7$$

$$+ \tfrac{1}{9}x^8$$

(14)

$$1 - 4x - 2x^2 + 4x^3 - 2x^4$$

$$4x^2 + 4x^3 - 8x^4 + 4x^5$$

$$x^4 - 4x^5 + 2x^6$$

$$4x^6 - 4x^7$$

$$+ x^8$$

Exercise XLI.

(3)

$$
\begin{array}{rl}
 & 4x^4 + 12x^3 + 5x^2 - 6x + 1\ (2x^2 + 3x - 1 \\
 & 4x^4 \\
4x^2 + 3x) & 12x^3 + 5x^2 \\
 & 12x^3 + 9x^2 \\
4x^2 + 6x - 1) & -4x^2 - 6x + 1 \\
 & -4x^2 - 6x + 1
\end{array}
$$

(4)

This may be worked by the rule or it may be bracketed so as to show the sq. root: thus $x^4 - 2x^2(y^2 + 1) + (y^2 + 1)^2$

(6)

$$
\begin{array}{rl}
 & 9a^4 + 12a^3 + 34a^2 + 20a + 25(3a^2 + 2a + 5 \\
 & 9a^4 \\
6a^2 + 2a) & 12a^3 + 34a^2 \\
 & 12a^3 + 4a^2 \\
6a^2 + 4a + 5) & 30a^2 + 20a + 25 \\
 & 30a^2 + 20a + 25
\end{array}
$$

(12)

$(x - y)^4 - 2(x^2 + y^2)(x - y)^2 + 2(x^4 + y^4) = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 - (2x^4 - 4x^3y + 4x^2y^2 - 4xy^3 + 2y^4) + 2x^4 + 2y^4 = x^4 + 2x^2y^2 + y^4$; and $\sqrt{x^4 + 2x^2y^2 + y^4} = x^2 + y^2$

(13) $(a^2 - b^2 + c^2 - d^2)$

$$
\begin{array}{rl}
 & a^4 - 2a^2b^2 + b^4 + 2a^2c^2 - 2b^2c^2 + c^4 - 2a^2d^2 + 2b^2d^2 - 2c^2d^2 + d^4 \\
 & a^4 \\
2a^2 - b^2) & -2a^2b^2 + b^4 \\
 & -2a^2b^2 + b^4 \\
2a^2 - 2b^2 + c^2) & 2a^2c^2 - 2b^2c^2 + c^4 \\
 & 2a^2c^2 - 2b^2c^2 + c^4 \\
2a^2 - 2b^2 + 2c^2 - d^2) & -2a^2d^2 + 2b^2d^2 - 2c^2d^2 + d^4 \\
 & -2a^2d^2 + 2b^2d^2 - 2c^2d^2 + d^4
\end{array}
$$

(14)

$$1-\tfrac{2}{3}x+\tfrac{10}{9}x^2-\tfrac{7}{6}x^3+\tfrac{7}{6}x^4-\tfrac{1}{2}x^5+\tfrac{1}{16}x^6\big(1-\tfrac{1}{3}x+x^2-\tfrac{1}{4}x^3$$
$$1$$
$$2-\tfrac{1}{3}x\big)\ \ -\tfrac{2}{3}x+\tfrac{10}{9}x^2$$
$$-\tfrac{2}{3}x+\tfrac{1}{9}x^2$$
$$2-\tfrac{2}{3}x+x^2\big)\ \ 2x^2-\tfrac{7}{6}x^3+\tfrac{7}{6}x^4$$
$$2x^2-\tfrac{4}{6}x^3+x^4$$
$$2-\tfrac{2}{3}x+2x^2-\tfrac{1}{4}x^3\big)-\tfrac{1}{2}x^3+\tfrac{1}{6}x^4-\tfrac{1}{2}x^5+\tfrac{1}{16}x^6$$
$$-\tfrac{1}{2}x^3+\tfrac{1}{6}x^4-\tfrac{1}{2}x^5+\tfrac{1}{16}x^6$$

(15)

$$\tfrac{1}{4}x^4+\frac{x^3}{y}+\frac{x^2}{y^2}-xy-2+\frac{y^2}{x^2}\left(\tfrac{1}{2}x^2+\frac{x}{y}-\frac{y}{x}\right.$$
$$\tfrac{1}{4}x^4$$
$$x^2+\frac{x}{y}\Big)\ \ \frac{x^3}{y}+\frac{x^2}{y^2}$$
$$\frac{x^3}{y}+\frac{x^2}{y^2}$$
$$x^2+\frac{2x}{y}-\frac{y}{x}\Big)\ -xy-2+\frac{y^2}{x^2}$$
$$-xy-2+\frac{y^2}{x^2}$$

Exercise XLII.

(2)

$$a^6+6a^5-40a^3+96a-64\big(a^2+2a-4$$
$$a^6$$
$$\begin{array}{l|l}
3a^4\ +6a^3\ +4a^2 & 6a^5-40a^3 \\
3a^4+6a^3+4a^2 & 6a^5+12a^4+8a^3 \\
\hline
3a^4+12a^3+12a^2 & -12a^4-48a^3+96a-64 \\
\qquad -12a^2-24a & \\
\qquad\qquad +16 & \\
3a^4+12a^3-24a+16 & -12a^4-48a^3+96a-64
\end{array}$$

(4)

$$a^6 - 6a^5 + 15a^4 - 20a^3 + 15a^2 - 6a + 1(a^2 - 2a + 1$$
$$a^6$$

$3a^4$ $\quad -6a^5 + 15a^4 - 20a^3$

$-6a^3$

$+4a^2$

$3a^4 - 6a^3 + 4a^2$ $\quad -6a^5 + 12a^4 - 8a^3$

$3a^4 - 12a^3 + 12a^2$ $\quad 3a^4 - 12a^3 + 15a^2 - 6a + 1$

$3a^2 - 6a$

$+1$

$3a^4 - 12a^3 + 15a^2 - 6a + 1$ $\quad 3a^4 - 12a^3 + 15a^2 - 6a + 1$

(6)

$$8x^6 - 36ax^5 + 102a^2x^4 - 171a^3x^3 + 204a^4x^2 - 144a^5x + 64a^6 \,|\, 2x^2 - 3ax + 4a^2$$
$$8x^6$$

$12x^4$ $\quad -36ax^5 + 102a^2x^4 - 171a^3x^3$

$-18ax^3$

$+9a^2x^2$

$12x^4 - 18ax^3 + 9a^2x^2$ $\quad -36ax^5 + 54a^2x^4 - 27a^3x^3$

$12x^4 - 36ax^3 + 27a^2x^2$ $\quad 48a^2x^4 - 144a^3x^3 + 204a^4x^2 - 144a^5x + 64a^6$

$24a^2x^2 - 36a^3x$

$+16a^4$

$12x^4 - 36ax^3 + 51a^2x^2 - 36a^3x + 16a^4$ $\quad 48a^2x^4 - 144a^3x^3 - 204a^4x^2 - 144a^5x + 64a^6$

(8)

$$a^3 + 3a^2b + 3ab^2 + b^3 + 3(a+b)^2c + 3(a+b)c^2 + \&c. \;(a + b + c + d + e$$
$$a^3$$

$3a^2 + 3ab + b^2$ $\quad 3a^2b + 3ab^2 + b^3$

$3a^2 + 3ab + b^2$ $\quad 3a^2b + 3ab^2 + b^3$

$3(a+b)^2 + 3(a+b)c + c^2$ $\quad 3(a+b)^2c + 3(a+b)c^2 + c^3$

$3(a+b)^2 + 3(a+b)c + c^2$ $\quad 3(a+b)^2c + 3(a+b)c^2 + c^3$

$3(a+b+c)^2 + 3(a+b+c)d + d^2$ $\quad 3(a+b+c)^2d + \&c.$

$3(a+b+c)^2 + 3(a+b+c)d + d^2$ $\quad 3(a+b+c)^2d + \&c.$

Exercise XLIII.

5. $\left\{\left(\dfrac{a^{-1}}{b^{-2}}\right)^{-n}\right\}^{-m} = \left(\dfrac{a^{n}}{b^{2n}}\right)^{-m}$; or $= \left(\dfrac{a^{-1}}{b^{-2}}\right)^{mn} = \left(\dfrac{b^{2}}{a}\right)^{mn}$

6. $\left(a^{-\frac{3}{6}} \times a^{-\frac{2}{3}}\right)^{-3} = \left(a^{-\frac{3}{5}-\frac{2}{3}}\right)^{-3} = \left(a^{-\frac{19}{15}}\right)^{-3} = a^{\frac{19}{5}} = \sqrt[5]{a^{19}}$

$\left(a^{\frac{1}{2}} \times a^{-\frac{2}{3}} \times a^{\frac{3}{4}}\right)^{-\frac{7}{3}} = \left(a^{\frac{1}{2}-\frac{2}{3}+\frac{3}{4}}\right)^{-\frac{7}{3}} = a^{-\frac{49}{36}}$

$\left(a^{-3} \times a^{\frac{1}{2}} \times a^{\frac{1}{3}} \times a^{\frac{1}{4}}\right)^{\frac{6}{23}} = \left(a^{-3+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}\right)^{\frac{6}{23}} = \left(a^{-\frac{3}{1}+\frac{13}{12}}\right)^{\frac{6}{23}}$

$= \left(a^{-\frac{23}{12}}\right)^{\frac{6}{23}} = a^{-\frac{1}{2}} = \dfrac{1}{\sqrt{a}}$

7. $\left(\sqrt[3]{\left\{\sqrt{\left(a^{-3}c^{-\frac{1}{2}}\right)ac}\right\}}\right)^{12} = \left(\left\{\left(a^{-3}c^{-\frac{1}{2}}\right)^{\frac{1}{2}}ac\right\}^{\frac{1}{3}}\right)^{12}$

$= \left(a^{-\frac{3}{2}}c^{-\frac{1}{4}}ac\right)^{4} = \left(a^{-\frac{3}{2}+1}c^{-\frac{1}{4}+1}\right)^{4} = \left(a^{-\frac{1}{2}}c^{\frac{3}{4}}\right)^{4} = a^{-2}c^{3} = \dfrac{c^{3}}{a^{2}}$

$\left\{\left(\left\{a^{\frac{1}{2}}\right\}^{\frac{1}{2}}\right)^{\frac{1}{2}} \times a^{\frac{1}{4}}\right\}^{-2} = \left\{\left(a^{\frac{1}{4}}\right)^{\frac{1}{2}} \times a^{\frac{1}{4}}\right)^{-2} = \left\{a^{\frac{1}{8}} \times a^{\frac{1}{4}}\right\}^{-2} = \left(a^{\frac{3}{8}}\right)^{-2} = a^{-\frac{3}{4}}$

8. $\left\{\left(a^{3}b^{\frac{1}{2}}\right)^{\frac{1}{4}}\left(a^{\frac{3}{4}}b^{3}c^{3}\right)^{\frac{1}{4}} \times a^{3}b^{3}c^{3}\left(\{ab^{6}\}^{\frac{1}{2}}c^{6}\right)^{\frac{1}{2}} \times a^{-1}b^{-\frac{1}{2}}c^{-\frac{21}{4}}\right\}^{\frac{16}{3}}$

$= \left(a^{\frac{3}{4}}b^{\frac{1}{8}}\ a^{\frac{3}{16}}b^{\frac{3}{4}}c^{\frac{3}{4}}\ a^{3}b^{3}c^{3}\ a^{\frac{1}{4}}b^{\frac{3}{2}}c^{3}\ a^{-1}\ b^{-\frac{1}{2}}c^{-\frac{21}{4}}\right)^{\frac{16}{3}}$

$= a^{\frac{3}{4}+\frac{3}{16}+3+\frac{1}{4}-1}\ b^{\frac{1}{8}+\frac{3}{4}+3+\frac{3}{2}-\frac{1}{2}}\ c^{\frac{3}{4}+3+3-\frac{21}{4}}\Big)^{\frac{16}{3}} = a^{17}b^{26}c^{8}$

9. $\dfrac{\left(x^{\frac{n}{r}} \times x^{\frac{m}{s}} \times x^{\frac{p}{t}}\right)^{rst}}{\left(y^{\frac{n}{s}} \times y^{\frac{r}{q}} \times y^{\frac{7}{4}}\right)^{4sq}} \div \dfrac{x^{mtrq}}{y^{4srq}} = \dfrac{\left(x^{\frac{n}{r}+\frac{m}{s}+\frac{p}{t}}\right)^{rst}}{\left(y^{\frac{n}{s}+\frac{r}{q}+\frac{7}{4}}\right)^{4sq}} \times \dfrac{y^{4srq}}{x^{mtrq}}$

$= \dfrac{\left(x^{\frac{nst+mrt+prs}{rst}}\right)^{rst}}{\left(y^{\frac{4nq+4rs+7sq}{4sq}}\right)^{4sq}} \times \dfrac{y^{4srq}}{x^{mtrq}} = \dfrac{x^{nst+mrt+prs}}{x^{mtrq}} \times \dfrac{y^{4srq}}{y^{4nq+4rs+7sq}}$

$= x^{nst+mrt+prs-mtrq}y^{4srq-4nq-4rs-7sq}$

$= x^{mtr(1-q)+(nt+pr)s}y^{4rs(q-1)-q(4n+7s)}$

(10)

$a^{\frac{3}{2}} - 3ab^{\frac{1}{2}} + 3a^{\frac{1}{2}}b - b^{\frac{3}{2}}$

$a^{\frac{1}{2}} - \quad b^{\frac{1}{2}}$

$a^{2} - 3a^{\frac{3}{2}}b^{\frac{1}{2}} + 3ab - a^{\frac{1}{2}}b^{\frac{3}{2}}$

$\quad - a^{\frac{3}{2}}b^{\frac{1}{2}} + 3ab - 3a^{\frac{1}{2}}b^{\frac{3}{2}} + b^{2}$

$a^{2} - 4a^{\frac{3}{2}}b^{\frac{1}{2}} + 6ab - 4a^{\frac{1}{2}}b^{\frac{3}{2}} + b^{2}$

(11)

$\left(a^{\frac{2}{3}} + x^{\frac{2}{3}}\right) - \left(ax\right)^{\frac{1}{3}}$

$\left(a^{\frac{2}{3}} + x^{\frac{2}{3}}\right) + \left(ax\right)^{\frac{1}{3}}$

$\left(a^{\frac{2}{3}} + x^{\frac{2}{3}}\right)^{2} - \left(ax\right)^{\frac{2}{3}}$

$= a^{\frac{4}{3}} + 2a^{\frac{2}{3}}x^{\frac{2}{3}} + x^{\frac{4}{3}} - a^{\frac{2}{3}}x^{\frac{2}{3}}$

$= a^{\frac{4}{3}} + \quad a^{\frac{2}{3}}x^{\frac{2}{3}} + x^{\frac{4}{3}}$

(12)

$$4x - 2x^{\frac12}y^{-\frac12} + 2x^{\frac12}z^{\frac13} - y^{-1} + y^{-\frac12}z^{\frac13}$$
$$2x^{\frac12} + y^{-\frac12} - z^{\frac13}$$
$$8x^{\frac32} - 4xy^{-\frac12} + 4xz^{\frac13} - 2x^{\frac12}y^{-1} + 2x^{\frac12}y^{-\frac12}z^{\frac13}$$
$$4xy^{-\frac12} - 2x^{\frac12}y^{-1} + 2x^{\frac12}y^{-\frac12}z^{\frac13} - y^{-\frac32} + y^{-1}z^{\frac13}$$
$$-4xz^{\frac13} + 2x^{\frac12}y^{-\frac12}z^{\frac13} - 2x^{\frac12}z^{\frac23} + y^{-1}z^{\frac13} - y^{-\frac12}z^{\frac23}$$
$$8x^{\frac32} - 4x^{\frac12}y^{-1} + 6x^{\frac12}y^{-\frac12}z^{\frac13} + 2y^{-1}z^{\frac13} - y^{-\frac32} - 2x^{\frac12}z^{\frac23} - y^{-\frac12}z^{\frac23}$$

(13)

$$-3x^{-4}y - 2x^{-3}\,)\;9x^{-9}y - 4x^{-7}y^{-1}\;(\,-3x^{-5} + 2x^{-4}y^{-1}$$
$$9x^{-9}y + 6x^{-8}$$
$$-6x^{-8} - 4x^{-7}y^{-1}$$
$$-6x^{-8} - 4x^{-7}y^{-1}$$

(14)

$$a^{\frac58} + a^{\frac12}b^{-\frac18} + a^{\frac38}b^{-\frac14} + a^{\frac14}b^{-\frac38} + a^{\frac18}b^{-\frac12} + b^{-\frac58}\,)$$
$$a + a^{\frac34}b^{-\frac14} - a^{\frac14}b^{-\frac34} - b^{-1}\;(\,a^{\frac38} - a^{\frac14}b^{-\frac18} + a^{\frac18}b^{-\frac14} - b^{-\frac38}$$
$$a + a^{\frac78}b^{-\frac18} + a^{\frac68}b^{-\frac14} + a^{\frac58}b^{-\frac38} + a^{\frac12}b^{-\frac12} + a^{\frac38}b^{-\frac58}$$
$$-a^{\frac78}b^{-\frac18} - a^{\frac58}b^{-\frac38} - a^{\frac48}b^{-\frac48} - a^{\frac38}b^{-\frac58} - a^{\frac28}b^{-\frac68} - b^{-1}$$
$$-a^{\frac78}b^{-\frac18} - a^{\frac68}b^{-\frac14} - a^{\frac58}b^{-\frac38} - a^{\frac48}b^{-\frac48} - a^{\frac38}b^{-\frac58} - a^{\frac28}b^{-\frac68}$$
$$a^{\frac68}b^{-\frac28} - b^{-1}$$
$$a^{\frac68}b^{-\frac28} + a^{\frac58}b^{-\frac38} + a^{\frac48}b^{-\frac48} + a^{\frac38}b^{-\frac58} + a^{\frac28}b^{-\frac68} + a^{\frac18}b^{-\frac78}$$
$$-a^{\frac58}b^{-\frac38} - a^{\frac48}b^{-\frac48} - a^{\frac38}b^{-\frac58} - a^{\frac28}b^{-\frac68} - a^{\frac18}b^{-\frac78} - b^{-1}$$
$$-a^{\frac58}b^{-\frac38} - a^{\frac48}b^{-\frac48} - a^{\frac38}b^{-\frac58} - a^{\frac28}b^{-\frac68} - a^{\frac18}b^{-\frac78} - b^{-1}$$

(15)

$$x^{-\frac{1}{3}} + 1 + x^{\frac{1}{3}}\big)\, x^{-1} + x^{-\frac{1}{3}} - 1 + x^{\frac{1}{3}} + x\,\big(x^{-\frac{2}{3}} - x^{-\frac{1}{3}} + 1 - x^{\frac{1}{3}} + x^{\frac{2}{3}}$$

$$\underline{x^{-1} + x^{-\frac{2}{3}} + x^{-\frac{1}{3}}}$$

$$-x^{-\frac{2}{3}} - 1 + x^{\frac{1}{3}} + x$$

$$\underline{-x^{-\frac{2}{3}} - x^{-\frac{1}{3}} - 1}$$

$$x^{-\frac{1}{3}} + x^{\frac{1}{3}} + x$$

$$\underline{x^{-\frac{1}{3}} + 1 + x^{\frac{1}{3}}}$$

$$-1 + x$$

$$\underline{-1 - x^{\frac{1}{3}} - x^{\frac{2}{3}}}$$

$$x^{\frac{1}{3}} + x^{\frac{2}{3}} + x$$

$$\underline{x^{\frac{1}{3}} + x^{\frac{2}{3}} + x}$$

(16)

$$\left(a^{\frac{3}{2}} - a + a^{\frac{1}{2}} + 1 - a^{-\frac{1}{2}} - a^{-1} + a^{-\frac{3}{2}}\right)^2$$

$$= a^3 - 2a^{\frac{5}{2}} + 2a^2 + 2a^{\frac{3}{2}} - 2a - 2a^{\frac{1}{2}} + 2$$

$$a^2 - 2a^{\frac{3}{2}} - 2a + 2a^{\frac{1}{2}} + 2 - 2a^{-\frac{1}{2}}$$

$$a + 2a^{\frac{1}{2}} - 2 - 2a^{-\frac{1}{2}} + 2a^{-1}$$

$$1 - 2a^{-\frac{1}{2}} - 2a^{-1} + 2a^{-\frac{3}{2}}$$

$$a^{-1} + 2a^{-\frac{3}{2}} - 2a^{-2}$$

$$a^{-2} - 2a^{-\frac{5}{2}}$$

$$\underline{a^{-3}}$$

$$= a^3 - 2a^{\frac{5}{2}} + 3a^2 - 3a + 2a^{\frac{1}{2}} + 3 - 6a^{-\frac{1}{2}} + a^{-1} + 4a^{-\frac{3}{2}} - a^{-2} - 2a^{-\frac{5}{2}}$$
$$+ a^{-3}$$

(17)

$$a^{\frac{2}{3}} + 2a^{\frac{1}{3}} - 1 - 2a^{-\frac{1}{3}} + a^{-\frac{2}{3}}\big(a^{\frac{1}{3}} + 1 - a^{-\frac{1}{3}}$$

$$\underline{a^{\frac{2}{3}}}$$

$$2a^{\frac{1}{3}} + 1\big)\quad 2a^{\frac{1}{3}} - 1$$

$$\underline{2a^{\frac{1}{3}} + 1}$$

$$2a^{\frac{1}{3}} + 2 - a^{-\frac{1}{3}}\big) - 2 - 2a^{-\frac{1}{3}} + a^{-\frac{2}{3}}$$

$$\underline{-2 - 2a^{-\frac{1}{3}} + a^{-\frac{2}{3}}}$$

(18)

$$
\begin{array}{l}
\qquad\qquad\qquad\qquad\qquad\qquad \left(x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 3 - 2x^{-\frac{1}{3}} + x^{-\frac{2}{3}}\right. \\
\qquad\qquad x^{\frac{4}{3}} - 4x + 10x^{\frac{2}{3}} - 16x^{\frac{1}{3}} + 19 - 16x^{-\frac{1}{3}} + 10x^{-\frac{2}{3}} - 4x^{-1} + x^{-\frac{4}{3}} \\
\qquad\qquad x^{\frac{4}{3}} \\
\hline
2x - 2x^{\frac{1}{3}}\,\big)\; -4x + 10x^{\frac{2}{3}} \\
\qquad\qquad\quad -4x + \ \ 4x^{\frac{2}{3}} \\
\hline
2x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 3\,\big)\quad 6x^{\frac{2}{3}} - 16x^{\frac{1}{3}} + 19 \\
\qquad\qquad\qquad\qquad 6x^{\frac{2}{3}} - 12x^{\frac{1}{3}} + \ \ 9 \\
\hline
2x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 6 - 2x^{-\frac{1}{3}}\,\big)\; -4x^{\frac{1}{3}} + 10 - 16x^{-\frac{1}{3}} + 10x^{-\frac{2}{3}} \\
\qquad\qquad\qquad\qquad\qquad\quad -4x^{\frac{1}{3}} + \ \ 8 - 12x^{-\frac{1}{3}} + \ \ 4x^{-\frac{2}{3}} \\
\hline
2x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 6 - 4x^{-\frac{1}{3}} + x^{-\frac{2}{3}}\,\big)\quad 2 - \ \ 4x^{-\frac{1}{3}} + 6x^{-\frac{2}{3}} - 4x^{-1} + x^{-\frac{4}{3}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 2 - \ \ 4x^{-\frac{1}{3}} + 6x^{-\frac{2}{3}} - 4x^{-1} + x^{-\frac{4}{3}} \\
\hline
\end{array}
$$

(19)

$$
\begin{array}{l}
3x^{-\frac{2}{3}}y^{2}\quad 3 + x^{\frac{2}{3}}y^{-2}\,\big)\; x^{-1}y^{3} - 3x^{-\frac{1}{3}}y + 3x^{\frac{1}{3}}y^{-1} - xy^{-3}\,\big(x^{-\frac{1}{3}}y - x^{\frac{1}{3}}y^{-1} \\
\qquad\qquad\qquad\quad x^{-1}y^{3} \\
\hline
\qquad\qquad\qquad\qquad -\ 3x^{-\frac{1}{3}}y + 3x^{\frac{1}{3}}y^{-1} - xy^{-3} \\
\qquad\qquad\qquad\qquad -\ 3x^{-\frac{1}{3}}y + 3x^{\frac{1}{3}}y^{-1} - xy^{-3} \\
\hline
\end{array}
$$

(20)

$$
\begin{array}{l|l}
 & \qquad\qquad\qquad\qquad\qquad \big|\, x^{\frac{2}{3}} - 2x^{\frac{1}{3}}y^{\frac{1}{6}} + 3y^{\frac{1}{3}} \\
 & x^{2} - 6x^{\frac{5}{3}}y^{\frac{1}{6}} + 21x^{\frac{4}{3}}y^{\frac{1}{3}} - 44xy^{\frac{1}{2}} + 63x^{\frac{2}{3}}y^{\frac{2}{3}} - 54x^{\frac{1}{3}}y^{\frac{5}{6}} + 27y \\
 & x^{2} \\
\hline
3x^{\frac{4}{3}} & -6x^{\frac{5}{3}}y^{\frac{1}{6}} + 21x^{\frac{4}{3}}y^{\frac{1}{3}} - 44xy^{\frac{1}{2}} \\
\quad -6xy^{\frac{1}{6}} & \\
\qquad +4x^{\frac{2}{3}}y^{\frac{1}{3}} & \\
\hline
3x^{\frac{4}{3}} - 6xy^{\frac{1}{6}} + 4x^{\frac{2}{3}}y^{\frac{1}{3}} & -6x^{\frac{5}{3}}y^{\frac{1}{6}} + 12x^{\frac{4}{3}}y^{\frac{1}{3}} - \ \ 8xy^{\frac{1}{2}} \\
\hline
3x^{\frac{4}{3}} - 12xy^{\frac{1}{6}} + 12x^{\frac{2}{3}}y^{\frac{1}{3}} & 9x^{\frac{4}{3}}y^{\frac{1}{3}} - 36xy^{\frac{1}{2}} + 63x^{\frac{2}{3}}y^{\frac{2}{3}} - 54x^{\frac{1}{3}}y^{\frac{5}{6}} + 27y \\
\qquad\qquad 9x^{\frac{2}{3}}y^{\frac{1}{3}} - 18x^{\frac{1}{3}}y^{\frac{1}{2}} & \\
\qquad\qquad\qquad + 9y^{\frac{2}{3}} & \\
\hline
3x^{\frac{4}{3}} - 12xy^{\frac{1}{6}} + 21x^{\frac{2}{3}}y^{\frac{1}{3}} - 18x^{\frac{1}{3}}y^{\frac{1}{2}} & 9x^{\frac{4}{3}}y^{\frac{1}{3}} - 36xy^{\frac{1}{2}} + 63x^{\frac{2}{3}}y^{\frac{2}{3}} - 54x^{\frac{1}{3}}y^{\frac{5}{6}} + 27y \\
\qquad\qquad\qquad + 9y^{\frac{2}{3}} & \\
\hline
\end{array}
$$

Exercise XLIV.

1. $2^{\frac{2}{3}} = (2^2)^{\frac{1}{3}} = 4^{\frac{1}{3}}$; $7^{\frac{3}{2}} = (7^3)^{\frac{1}{2}} = 343^{\frac{1}{2}}$; $2^{\frac{4}{5}} = (2^4)^{\frac{1}{5}} = 16^{\frac{1}{5}}$; $(\frac{3}{2})^{\frac{2}{3}} = \{(\frac{3}{2})^2\}^{\frac{1}{3}} = (\frac{9}{4})^{\frac{1}{3}}$; $(\frac{13}{4})^{-\frac{2}{5}} = (\frac{4}{13})^{\frac{2}{5}} = (\frac{16}{169})^{\frac{1}{5}}$; $3^{\frac{2}{3}} = 9^{\frac{1}{3}}$; $(a^{\frac{5}{2}})^{-\frac{2}{3}}$

$$= \left\{\left(\frac{a}{1}\right)^{\frac{5}{2}}\right\}^{-\frac{2}{3}} = \left\{\left(\frac{1}{a}\right)^{\frac{5}{2}}\right\}^{\frac{2}{3}} = \left(\frac{1}{a}\right)^{\frac{5}{3}} = \left(\frac{1}{a^5}\right)^{\frac{1}{3}}$$

2. $a = a^{\frac{2}{2}} = (a^2)^{\frac{1}{2}}$; $3^{\frac{2}{2}} = 9^{\frac{1}{2}}$; $(\frac{9}{2})^{\frac{2}{2}} = (\frac{81}{4})^{\frac{1}{2}}$; $(2a)^{\frac{2}{2}} = (4a^2)^{\frac{1}{2}}$; $(3a^2b)^{\frac{2}{2}} = (9a^4b^2)^{\frac{1}{2}}$; $(4x^2y^3)^{\frac{2}{2}} = (16x^4y^6)^{\frac{1}{2}}$

$a = a^{\frac{3}{3}} = (a^{-3})^{-\frac{1}{3}} = \left(\dfrac{1}{a^3}\right)^{-\frac{1}{3}}$; $(3^{-3})^{-\frac{1}{3}} = (\frac{1}{27})^{-\frac{1}{3}}$; $\{(\frac{9}{2})^{-3}\}^{-\frac{1}{3}} = (\frac{8}{729})^{-\frac{1}{3}}$; $\{(2a)^{-3}\}^{-\frac{1}{3}} = \left(\dfrac{1}{8a^3}\right)^{-\frac{1}{3}}$; $\{(3a^2b)^{-3}\}^{-\frac{1}{3}} = \left(\dfrac{1}{27a^6b^3}\right)^{-\frac{1}{3}}$;

$\{(4x^2y^3)^{-3}\}^{-\frac{1}{3}} = \left(\dfrac{1}{64x^6y^9}\right)^{-\frac{1}{3}}$

$a = a^{\frac{4}{4}} = (a^4)^{\frac{1}{4}}$; $3^{\frac{4}{4}} = (81)^{\frac{1}{4}}$; $(\frac{9}{2})^{\frac{4}{4}} = (\frac{6561}{16})^{\frac{1}{4}}$; $(2a)^{\frac{4}{4}} = (16a^4)^{\frac{1}{4}}$; $(3a^2b)^{\frac{4}{4}} = (81a^8b^4)^{\frac{1}{4}}$; $(4x^2y^3)^{\frac{4}{4}} = (256x^8y^{12})^{\frac{1}{4}}$

3. $a^2 = a^{\frac{4}{2}} = (a^{-4})^{-\frac{1}{2}} = \left(\dfrac{1}{a^4}\right)^{-\frac{1}{2}}$; $3^{\frac{1}{2}} = (3^{-1})^{-\frac{1}{2}} = (\frac{1}{3})^{-\frac{1}{2}}$;

$\{(2a^2b^3)^{-2}\}^{-\frac{1}{2}} = \left(\dfrac{1}{4a^4b^6}\right)^{-\frac{1}{2}}$; $\{(ac^2)^{\ 2}\}^{-\frac{1}{2}} = \left(\dfrac{1}{a^2c^4}\right)^{-\frac{1}{2}}$; $\{(\frac{22}{5})^{-2}\}^{-\frac{1}{2}} = (\frac{25}{484})^{-\frac{1}{2}}$; $3^{-2} = \frac{1}{9} = \{(\frac{1}{9})^{-2}\}^{-\frac{1}{2}} = 81^{-\frac{1}{2}}$; $(\frac{7}{4})^{-3} = (\frac{4}{7})^3 = \frac{64}{343} = \{(\frac{64}{343})^{-2}\}^{-\frac{1}{2}} = \left(\dfrac{343^2}{64^2}\right)^{-\frac{1}{2}} = (\frac{117649}{4096})^{-\frac{1}{2}}$; $(x^{-1}y^{-2}z^3)^{-1} = \left(\dfrac{z^3}{xy^2}\right)^{-1}$

$$= \left\{\left(\frac{z^3}{xy^2}\right)^2\right\}^{-\frac{1}{2}} = \left(\frac{z^6}{x^2y^4}\right)^{-\frac{1}{2}}$$

4. $(\frac{3}{4})^{\frac{2}{2}} \times (\frac{2}{3})^{\frac{1}{2}} = (\frac{9}{16} \times \frac{2}{3})^{\frac{1}{2}} = (\frac{3}{8})^{\frac{1}{2}}$; $\dfrac{1}{2}\left(\dfrac{a^2}{b}\right)^{-\frac{2}{3}} = \dfrac{1}{2}\left(\dfrac{b}{a^2}\right)^{\frac{2}{3}} = \left(\dfrac{1}{2}\right)^{\frac{3}{3}} \times \left(\dfrac{b^2}{a^4}\right)^{\frac{1}{3}} = \left(\dfrac{1}{8} \times \dfrac{b^2}{a^4}\right)^{\frac{1}{3}} = \left(\dfrac{b^2}{8a^4}\right)^{\frac{1}{3}}$

5. $\dfrac{2}{3}\left(\dfrac{3ab}{9}\right)^{\frac{1}{2}} = \frac{2}{3} \times \frac{1}{3}\sqrt{3ab} = \frac{2}{9}\sqrt{3ab}$; $\dfrac{a}{b}\left(\dfrac{6}{8}\right)^{\frac{3}{4}} = \dfrac{a}{2b}\sqrt[3]{6}$;

$\frac{2}{3}(\frac{7}{2})^{\frac{1}{2}} = \frac{2}{3}(\frac{14}{4})^{\frac{1}{2}} = \frac{2}{3} \times \frac{1}{2}\sqrt{14} = \frac{1}{3}\sqrt{14}$; $\frac{4}{5}(\frac{4}{25})^{\frac{1}{3}} = \frac{4}{5}(\frac{20}{125})^{\frac{1}{3}} = \frac{4}{25}\sqrt[3]{20}$;

$$\frac{3a}{4}\left(\frac{b}{2}\right)^{-\frac{2}{3}} = \frac{3a}{4}\left(\frac{2}{b}\right)^{\frac{2}{3}} = \frac{3a}{4}\left(\frac{4}{b^2}\right)^{\frac{1}{3}} = \frac{3a}{4}\left(\frac{4b}{b^3}\right)^{\frac{1}{3}} = \frac{3a}{4b}\sqrt[3]{4b}$$

6. $2a\left(\dfrac{3a^2}{5}\right)^{-\frac{2}{3}} = 2a\left(\dfrac{9a^4}{25}\right)^{-\frac{1}{3}} = 2a\left(\dfrac{25}{9a^4}\right)^{\frac{1}{3}} = \left(8a^3 \times \dfrac{25}{9a^4}\right)^{\frac{1}{3}}$

$= \left(\dfrac{200}{9a}\right)^{\frac{1}{3}}$, or $\left(\dfrac{9a}{200}\right)^{-\frac{1}{3}}$

$\dfrac{2}{5}\left(\dfrac{3m}{4}\right)^{\frac{2}{5}} = \dfrac{2}{5}\left(\dfrac{9m^2}{16}\right)^{\frac{1}{5}} = \left(\dfrac{32}{3125} \times \dfrac{9m^2}{16}\right)^{\frac{1}{5}} = \left(\dfrac{18m^2}{3125}\right)^{\frac{1}{5}}$

$(am + pq)^{\frac{2}{2}} \times \left(\dfrac{am - pq}{am + pq}\right)^{\frac{1}{2}} = \left\{(am + pq)(am + pq) \times \dfrac{am - pq}{am + pq}\right\}^{\frac{1}{2}}$

$= (a^2m^2 - p^2q^2)^{\frac{1}{2}}$, or $\left(\dfrac{1}{a^2m^2 - p^2q^2}\right)^{-\frac{1}{2}}$

7. $(135)^{\frac{1}{3}} = (27 \times 5)^{\frac{1}{3}} = 3\sqrt[3]{5}$; $\sqrt{162} = \sqrt{81 \times 2} = 9\sqrt{2}$;

$\sqrt[4]{80} = \sqrt[4]{16 \times 5} = 2\sqrt[4]{5}$; $7\sqrt[3]{324} = 7\sqrt[3]{27 \times 12} = 7 \times 3\sqrt[3]{12} = 21\sqrt[3]{12}$;

$\frac{1}{2}\sqrt{\frac{3}{7}} = \frac{1}{2}\sqrt{\frac{21}{49}} = \frac{1}{14}\sqrt{21}$;

$(\frac{1}{64})^{\frac{1}{6}}\left(\dfrac{704m^5}{11a}\right)^{\frac{1}{6}} = \left(\dfrac{704m^5}{704a}\right)^{\frac{1}{6}} = \left(\dfrac{m^5}{a}\right)^{\frac{1}{6}} = \dfrac{1}{a}(a^5m^5)^{\frac{1}{6}}$

8. $\left\{\dfrac{ab^2}{6(a + x)}\right\}^{\frac{1}{2}} = \left\{\dfrac{b^2 \times 6a(a + x)}{36(a + x)^2}\right\}^{\frac{1}{2}} = \left\{\dfrac{b^2}{36(a + x)^2} \times 6a(a + x)\right\}^{\frac{1}{2}}$

$= \dfrac{b}{6(a + x)}\sqrt{6a(a + x)}$;

$\dfrac{a}{b}\sqrt{\left(\dfrac{c^2m^2}{a^2} \times \dfrac{1}{n}\right)} = \dfrac{a}{b} \times \dfrac{cm}{a}\sqrt{\dfrac{1}{n}} = \dfrac{cm}{b}\sqrt{\dfrac{n}{n^2}} = \dfrac{cm}{bn}\sqrt{n}$;

$\sqrt[m]{(a^{m+n}x)} = a\sqrt[m]{a^nx}$; $\sqrt[q]{\left\{\dfrac{(az - z^2)^{2q}(b + z)}{c + z}\right\}} = (az - z^2)^2 \times$

$\sqrt[q]{\left(\dfrac{b + z}{c + z}\right)} = (az - z^2)^2 \times \sqrt[q]{\left\{\dfrac{(b + z)(c + z)^{q-1}}{(c + z)^q}\right\}}$

$= \dfrac{(az - z^2)^2}{c + z}\sqrt[q]{\{(b + z)(c + z)^{q-1}\}}$

$3\sqrt{2}$ and $3\sqrt[3]{3}$, or $18^{\frac{1}{2}}$ and $81^{\frac{1}{3}}$, or $18^{\frac{3}{6}}$ and $81^{\frac{2}{6}}$; or $(5832)^{\frac{1}{6}}$ and $(6561)^{\frac{1}{6}}$

or $(\frac{297}{5})^{\frac{1}{3}}$; $(44)^{\frac{1}{2}}$ and $(5103)^{\frac{1}{6}}$; or $(\frac{88209}{25})^{\frac{1}{6}}$, $(85184)^{\frac{1}{6}}$ and $(5103)^{\frac{1}{6}}$,

or $(3528\frac{9}{25})^{\frac{1}{6}}$, $(85184)^{\frac{1}{6}}$ and $(5103)^{\frac{1}{6}}$

10. $12\sqrt{2}+12\sqrt{2}-\sqrt{2}-8\sqrt{2}+35\sqrt{2}=59\sqrt{2}-9\sqrt{2}=50\sqrt{2}$;
$8\sqrt{(\tfrac{3}{4})}+4\sqrt{15}-\tfrac{11}{5}\sqrt{15}+\sqrt{\tfrac{15}{25}}=4\sqrt{3}+4\sqrt{15}-\tfrac{11}{5}\sqrt{15}+\tfrac{1}{5}\sqrt{15}$
$=4\sqrt{3}+4\sqrt{15}-2\sqrt{15}=4\sqrt{3}+2\sqrt{15}$

11. $2\sqrt{7}+6\sqrt{7}+3\sqrt[3]{3}-4\sqrt[3]{3}=8\sqrt{7}-\sqrt[3]{3}$;
$$3ab^2\sqrt{ac}+2a^2\sqrt{ac}-\frac{c^4}{b}\sqrt{ac}=\left(3ab^2+2a^2-\frac{c^4}{b}\right)\sqrt{ac}$$

12. $\sqrt[m]{2^m a^{mp} b^{mn}}\times\sqrt[m]{a^3b^5}+\sqrt[m]{3^m a^{3m-mn}b^m}\times\sqrt[m]{a^3b^5}-\sqrt[m]{c^{2m}}\times\sqrt[m]{a^3b^5}$
$=2a^pb^n\sqrt[m]{a^3b^5}+3a^{3-n}b\sqrt[m]{a^3b^5}-c^2\sqrt[m]{a^3b^5}=(2a^pb^n+3a^{3-n}b-c^2)\sqrt[m]{a^3b^5}$

13. $6\sqrt{200}=6\sqrt{100\times 2}=60\sqrt{2}$; $35\sqrt{60}=35\sqrt{4\times 15}=70\sqrt{15}$;
$(3\times 6^{\frac{3}{6}})\times(4\times 60^{\frac{2}{6}})=(3\times 4)\times(216^{\frac{1}{6}}\times 3600^{\frac{1}{6}}$
$=12\sqrt[6]{64\times 12150}=24\sqrt[6]{12150}$

14. $16^{\frac{2}{6}}\times 8^{\frac{3}{6}}=\sqrt[6]{16\times 16\times 8\times 8\times 8}=\sqrt[6]{4^6\times 32}=4\sqrt[6]{32}$;
$28a^{\frac{3}{6}}\times a^{\frac{4}{6}}=28a^{\frac{7}{6}}=28a\sqrt[6]{a}$; $2\times 3^{\frac{3}{6}}\times 72^{\frac{1}{6}}=2\times(27\times 72)^{\frac{1}{6}}$
$=2\sqrt[6]{1944}$; $7\sqrt[3]{24}\times\tfrac{1}{2}\sqrt[3]{5}=14\sqrt[3]{3}\times\tfrac{1}{2}\sqrt[3]{5}=7\sqrt[3]{15}$

15. $\dfrac{ax}{bc}\times\dfrac{by}{cd}\times\dfrac{c^2d}{a}\times\{(ax)^{\frac{1}{2}}\times(by)^{\frac{1}{3}}\times(cz)^{\frac{1}{4}}\}$
$=xy\{(ax)^{\frac{6}{12}}\times(by)^{\frac{4}{12}}\times(cz)^{\frac{3}{12}}\}=xy(a^6b^4c^3x^6y^4z^3)^{\frac{1}{12}}$
$(x-\sqrt{xy}+y)(\sqrt{x}+\sqrt{y})=x^{\frac{3}{2}}+y^{\frac{3}{2}}$ [See Alg. Art. 179.]

16. $(2\sqrt{3}+\tfrac{2}{3}\sqrt{\tfrac{3}{5}})(3\sqrt{2\tfrac{1}{2}}-4\sqrt{3})=(2\sqrt{3}+\tfrac{2}{15}\sqrt{15})(\tfrac{3}{2}\sqrt{10}-4\sqrt{3})$
$=3\sqrt{30}+\tfrac{1}{5}\sqrt{150}-8\sqrt{9}-\tfrac{8}{15}\sqrt{45}=3\sqrt{30}+\sqrt{6}-24-\tfrac{8}{5}\sqrt{5}$

17. $\tfrac{3}{4}\sqrt{\tfrac{2}{3}}=\tfrac{3}{4}(\tfrac{6}{9})^{\frac{1}{2}}=\tfrac{1}{4}\sqrt{6}$; $\tfrac{5}{3}(\tfrac{7}{8})^{\frac{1}{2}}=\tfrac{5}{3}(\tfrac{14}{16})^{\frac{1}{2}}=\tfrac{5}{12}\sqrt{14}$;
$2(\tfrac{2}{3}\div\tfrac{3}{5})^{\frac{1}{2}}=2\sqrt{\tfrac{10}{9}}=\tfrac{2}{3}\sqrt{10}$; $\tfrac{2}{3}(\tfrac{5}{2}\div\tfrac{13}{4})^{\frac{1}{2}}=\tfrac{2}{3}\sqrt{\tfrac{10}{13}}=\tfrac{2}{3}\sqrt{\tfrac{130}{169}}=\tfrac{2}{39}\sqrt{130}$

18. $2(12^{\frac{3}{6}}\div 7^{\frac{2}{6}})=2\times\left(\dfrac{1728}{49}\right)^{\frac{1}{6}}=2\times\left(\dfrac{64\times 27}{49}\right)^{\frac{1}{6}}=4\times\left(\dfrac{27}{49}\right)^{\frac{1}{6}}$
$=4\times\left(\dfrac{27\times 7^4}{7^6}\right)^{\frac{1}{6}}=\tfrac{4}{7}\sqrt[6]{64827}$

$$\frac{3}{2}\left(\frac{\sqrt[3]{4}}{\sqrt{5}}\right)=\frac{3}{2}\left(\frac{\sqrt[6]{16}}{\sqrt[6]{125}}\right)=\frac{3}{2}\left(\frac{16}{125}\right)^{\frac{1}{6}}=\frac{3}{2}\left(\frac{16\times 125}{5^6}\right)^{\frac{1}{6}}=\frac{3}{10}\sqrt[6]{2000}$$

$$\frac{4}{3}\left(\frac{\sqrt{\frac{2}{3}}}{\sqrt[3]{\frac{3}{2}}}\right)=\frac{4}{3}\left(\frac{\sqrt[6]{\frac{8}{27}}}{\sqrt[6]{\frac{9}{4}}}\right)=\frac{4}{3}\left(\frac{\frac{8}{27}}{\frac{9}{4}}\right)^{\frac{1}{6}}=\frac{4}{3}\left(\frac{32}{243}\right)^{\frac{1}{6}}=\frac{4}{3}\left(\frac{96}{729}\right)^{\frac{1}{6}}=\frac{4}{9}\sqrt[6]{96}$$

$$\frac{4}{3}\left(\frac{\sqrt[3]{ax}}{\sqrt{ax}}\right)=\frac{4}{3}\left(\frac{\sqrt[6]{a^2x^2}}{\sqrt[6]{a^3x^3}}\right)=\frac{4}{3}\left(\frac{a^2x^2}{a^3x^3}\right)^{\frac{1}{6}}=\frac{4}{3}\left(\frac{a^5x^5}{a^6x^6}\right)^{\frac{1}{6}}=\frac{4}{3ax}\sqrt[6]{a^5x^5}$$

19. $\dfrac{\sqrt{2}}{\frac{1}{4}\sqrt{2}} + \dfrac{3\sqrt{\frac{1}{2}}}{\frac{1}{4}\sqrt{2}} = 4 + \dfrac{\frac{3}{2}\sqrt{2}}{\frac{1}{4}\sqrt{2}} = 4 + \dfrac{\frac{3}{2}}{\frac{1}{4}} = 4 + 6 = 10;$

$$\frac{4\sqrt{3}}{2\sqrt[4]{3}} - \frac{5\sqrt[3]{4}}{2\sqrt[4]{3}} + \frac{6\sqrt[6]{7}}{2\sqrt[4]{3}} = \frac{4\sqrt[4]{9}}{2\sqrt[4]{3}} - \frac{5\sqrt[12]{256}}{2\sqrt[12]{27}} + \frac{6\sqrt[12]{49}}{2\sqrt[12]{27}}$$

$$= 2\sqrt[4]{3} - \tfrac{5}{2}\left(\tfrac{256}{27}\right)^{\frac{1}{12}} + 3\left(\tfrac{49}{27}\right)^{\frac{1}{12}} = 2\sqrt[4]{3} - \frac{5}{6}\sqrt[12]{5038848} + \sqrt[12]{964467};$$

$$\left(\frac{\dfrac{ab^{n-1}c^2}{a^3d^{-1}}}{c^{n-1}b^{-2}}\right)^{\frac{1}{5}} = \left(\frac{\dfrac{ab^{n-1}c^2}{a^3b^2}}{c^{n-1}d}\right)^{\frac{1}{5}} = \left(\frac{ab^{n-1}c^{n+1}d}{a^3b^2}\right)^{\frac{1}{5}}$$

$$= \left(\frac{a^3b^{n-3}c^{n+1}d}{a^5}\right)^{\frac{1}{5}} = \&c.$$

20. Multiplying by $\sqrt{7} - 6$ we have $(\sqrt{7})^2 - 6^2 = 7 - 36 = -29$

Multiplying by $\sqrt{3} + \sqrt{2}$ we have $(\sqrt{3})^2 - (\sqrt{2})^2 = 3 - 2 = 1$

Multiplying by $4\sqrt{3} + 6\sqrt{2\frac{1}{2}}$ we have $(4\sqrt{3})^2 - (6\sqrt{2\frac{1}{2}})^2 = 48 - 90 = -42$

Multiplying by $\frac{7}{3}\sqrt{\frac{1}{2}} - \frac{3}{5}\sqrt{2}$ we have $(\frac{7}{3}\sqrt{\frac{1}{2}})^2 - (\frac{3}{5}\sqrt{2})^2 = \frac{49}{18} - \frac{18}{25} = 2\frac{1}{450}$

Multiplying by $\frac{2}{3}\sqrt{\frac{1}{5}} + \frac{3}{4}\sqrt{\frac{1}{6}}$ we have $(\frac{2}{3}\sqrt{\frac{1}{5}})^2 - (\frac{3}{4}\sqrt{\frac{1}{6}})^2 = \frac{4}{45} - \frac{3}{32} = -\frac{7}{1440}$

21. $\dfrac{2(\sqrt{3} - 2\sqrt{5})}{(\sqrt{3} + 2\sqrt{5})(\sqrt{3} - 2\sqrt{5})} = \dfrac{2\sqrt{3} - 4\sqrt{5}}{(\sqrt{3})^2 - (2\sqrt{5})^2} = \dfrac{2\sqrt{3} - 4\sqrt{5}}{3 - 20} = \dfrac{4\sqrt{5} - 2\sqrt{3}}{17}$

$$\frac{(\sqrt{2} + \sqrt{3})(2\sqrt{5} + 3\sqrt{6})}{(2\sqrt{5} - 3\sqrt{6})(2\sqrt{5} + 3\sqrt{6})} = \frac{2\sqrt{10} + 2\sqrt{15} + 3\sqrt{12} + 3\sqrt{18}}{20 - 54}$$

$$= \frac{2\sqrt{10} + 2\sqrt{15} + 6\sqrt{3} + 9\sqrt{2}}{-34}$$

$$\frac{(2\sqrt{3} + \sqrt{11})(7\sqrt{8} + 8\sqrt{7})}{(7\sqrt{8} - 8\sqrt{7})(7\sqrt{8} + 8\sqrt{7})} = \frac{14\sqrt{24} + 7\sqrt{88} + 16\sqrt{21} + 8\sqrt{77}}{392 - 448}$$

$$= \frac{28\sqrt{6} + 14\sqrt{22} + 16\sqrt{21} + 8\sqrt{77}}{-56} = \&c.$$

22. $\dfrac{3(\sqrt{3} + \sqrt{x})}{(\sqrt{3} - \sqrt{x})(\sqrt{3} + \sqrt{x})} = \dfrac{3\sqrt{3} + 3\sqrt{x}}{3 - x};\ \dfrac{(a\sqrt{m} - m\sqrt{a})(a\sqrt{m} - m\sqrt{a})}{(a\sqrt{m} - m\sqrt{a})(a\sqrt{m} + m\sqrt{a})}$

$$= \frac{(a\sqrt{m} - m\sqrt{a})^2}{a^2m - m^2a} = \frac{a^2m - 2am\sqrt{ma} + m^2a}{a^2m - m^2a} = \frac{a - 2\sqrt{ma} + m}{a - m}$$

$$\frac{(2 + 3\sqrt{\frac{2}{3}})(\frac{1}{3}\sqrt{\frac{1}{2}} + \frac{2}{3}\sqrt{\frac{3}{5}})}{(\frac{1}{3}\sqrt{\frac{1}{2}} - \frac{2}{3}\sqrt{\frac{3}{5}})(\frac{1}{3}\sqrt{\frac{1}{2}} + \frac{2}{3}\sqrt{\frac{3}{5}})} = \frac{\frac{2}{3}\sqrt{\frac{1}{2}} + \frac{4}{3}\sqrt{\frac{3}{5}} + \sqrt{\frac{1}{3}} + 2\sqrt{\frac{2}{5}}}{\frac{1}{9} \times \frac{1}{2} - \frac{4}{9} \times \frac{3}{5}}$$

$$= \frac{\frac{1}{3}\sqrt{2} + \frac{4}{15}\sqrt{15} + \frac{1}{3}\sqrt{3} + \frac{2}{5}\sqrt{10}}{-\frac{19}{90}} = \&c$$

3. $\dfrac{(\sqrt{x^2+x+1}-\sqrt{x^2-x-1})(\sqrt{x^2+x+1}-\sqrt{x^2-x-1})}{(\sqrt{x^2+x+1}+\sqrt{x^2-x-1})(\sqrt{x^2+x+1}-\sqrt{x^2-x-1})}$

$$=\frac{(\sqrt{x^2+x+1}-\sqrt{x^2-x-1})^2}{(\sqrt{x^2+x+1})^2-(\sqrt{x^2-x-1})^2}$$

$$=\frac{x^2+x+1-2(\sqrt{x^2+x+1})(\sqrt{x^2-x-1})+x^2-x-1}{(x^2+x+1)-(x^2-x-1)}$$

$$=\frac{2x^2-2\sqrt{x^4-x^2-2x-1}}{2x+2}=\frac{x^2-\sqrt{x^4-x^2-2x-1}}{x+1}$$

24. $\dfrac{\sqrt{3}-\sqrt{2}-\sqrt{5}}{(\sqrt{3}-\sqrt{2}+\sqrt{5})(\sqrt{3}-\sqrt{2}-\sqrt{5})}$

$$=\frac{\sqrt{3}-\sqrt{2}-\sqrt{5}}{\{(\sqrt{3}-\sqrt{2})+\sqrt{5}\}\{(\sqrt{3}-\sqrt{2})-\sqrt{5}\}}=\frac{\sqrt{3}-\sqrt{2}-\sqrt{5}}{(\sqrt{3}-\sqrt{2})^2-5}$$

$$=\frac{\sqrt{3}-\sqrt{2}-\sqrt{5}}{3-2\sqrt{6}+2-5}=\frac{\sqrt{3}-\sqrt{2}-\sqrt{5}}{-2\sqrt{6}}=\frac{(\sqrt{3}-\sqrt{2}-\sqrt{5})\times\sqrt{6}}{-2\sqrt{6}\times\sqrt{6}}$$

$$=\frac{\sqrt{18}-\sqrt{12}-\sqrt{30}}{-12}=\frac{2\sqrt{3}+\sqrt{30}-3\sqrt{2}}{12}$$

$$\frac{(1-3\sqrt{2})(1+3\sqrt{2}+\sqrt{3})}{(1+3\sqrt{2}-\sqrt{3})(1+3\sqrt{2}+\sqrt{3})}=\frac{\sqrt{3}-3\sqrt{6}-17}{(1+3\sqrt{2})^2-3}=\frac{\sqrt{3}-3\sqrt{6}-17}{16+6\sqrt{2}}$$

$$=\frac{(\sqrt{3}-3\sqrt{6}-17)(16-6\sqrt{2})}{(16+6\sqrt{2})(16-6\sqrt{2})}=\frac{26\sqrt{3}-27\sqrt{6}+51\sqrt{2}-136}{92}$$

$$\frac{(2+3\sqrt{3})(1+2\sqrt{3}+\sqrt{2})}{(1+2\sqrt{3}-\sqrt{2})(1+2\sqrt{3}+\sqrt{2})}=\frac{20+7\sqrt{3}+2\sqrt{2}+3\sqrt{6}}{11+4\sqrt{3}}$$

$$\frac{(20+7\sqrt{3}+2\sqrt{2}+3\sqrt{6})(11-4\sqrt{3})}{(11+4\sqrt{3})(11-4\sqrt{3})}=\frac{136-3\sqrt{3}-14\sqrt{2}+25\sqrt{6}}{73}$$

Exercise XLV.

6. Let $\sqrt{42+3\sqrt{174\frac{2}{9}}}=\sqrt{x}+\sqrt{}$

Then $\sqrt{42-3\sqrt{174\frac{2}{9}}}=\sqrt{x}-\sqrt{y}$

Or $\sqrt{1764-1568}=\sqrt{196}=14=x-y$

Also $42+3\sqrt{174\frac{2}{9}}=x+2\sqrt{xy}+y$ or $42=x+y$

$\therefore 2x=56$ or $x=28$; $2y=28$ or $y=14$

$\therefore \sqrt{42+3\sqrt{174\frac{2}{9}}}=\sqrt{28}+\sqrt{14}=2\sqrt{7}+\sqrt{14}$

9. Let $\sqrt{a - 2\sqrt{a-1}} = \sqrt{x} - \sqrt{y}$ (I)

then $\sqrt{a + 2\sqrt{a-1}} = \sqrt{x} + \sqrt{y}$ (II);

or $\sqrt{a^2 - 4a + 4} = x - y$

or $a - 2 = x - y$

Squaring (I) we get $a = x + y \therefore 2a - 2 = 2x$, &c.

10. Let $\sqrt{2a + 2\sqrt{a^2 - b^2}} = \sqrt{x} + \sqrt{y}$

$\sqrt{2a - 2\sqrt{a^2 - b^2}} = \sqrt{x} - \sqrt{y}$

$\sqrt{4a^2 - 4a^2 + 4b^2} = \sqrt{4b^2} = 2b = x - y$

And $2a = x + y$

$2x = 2b + 2a$; or $x = a + b$

$2y = 2a - 2b$; or $y = a - b$

11. Let $\sqrt{8 + \sqrt{39}} = \sqrt{x} + \sqrt{y}$

Then $\sqrt{8 - \sqrt{39}} = \sqrt{x} - \sqrt{y}$

Or $\sqrt{64 - 39} = \sqrt{25} = 5 = x - y$

Also $8 = x + y$

$\therefore 2x = 13$ or $x = \frac{13}{2}$, and $2y = 3 \therefore y = \frac{3}{2}$

$\therefore \sqrt{8 + 39} = \sqrt{\frac{13}{2}} + \sqrt{\frac{3}{2}} = \frac{1}{2}\sqrt{26} + \frac{1}{2}\sqrt{3}$

12. Let $\sqrt{\frac{a^2}{4} + \frac{1}{2}b\sqrt{a^2 - b^2}} = \sqrt{x} + \sqrt{y}$

$\sqrt{\frac{a^2}{4} - \frac{1}{2}b\sqrt{a^2 - b^2}} = \sqrt{x} - \sqrt{y}$

$\sqrt{\frac{a^4}{16} - \frac{1}{4}b^2(a^2 - b^2)} = \sqrt{\frac{a^4}{16} - \frac{a^2b^2}{4} + \frac{b^4}{4}} = \frac{a^2}{4} - \frac{b^2}{2} = x - y$

And $\frac{a^2}{4} = x + y$

Then $2x = \frac{a^2}{2} - \frac{b^2}{2}$; or $x = \frac{a^2}{4} - \frac{b^2}{4}$

$2y = \frac{b^2}{2}$; or $y = \frac{b^2}{4} \therefore$ &c.

Exercise XLVI.

1. $\sqrt{32}-\sqrt{24}=\sqrt{8}(2-\sqrt{3})$ $\therefore$ $\sqrt{\sqrt{32}-\sqrt{24}}=\sqrt{\sqrt{8}(2-\sqrt{3})}$
$=\sqrt[4]{8}\sqrt{2-\sqrt{3}}=\sqrt[4]{8}(\frac{1}{2}\sqrt{6}-\frac{1}{2}\sqrt{2})=\sqrt[4]{8}(\frac{1}{2}\sqrt[4]{36}-\frac{1}{2}\sqrt[4]{4})=\frac{1}{2}(\sqrt[4]{288}-\sqrt[4]{32})$
$=\sqrt[4]{18}-\sqrt[4]{2}$

2. $3\sqrt{5}+\sqrt{40}=\sqrt{5}(3+2\sqrt{2})$ $\therefore$ $\sqrt{3\sqrt{5}+\sqrt{40}}=\sqrt{\sqrt{5}(3+2\sqrt{2})}$
$=\sqrt[4]{5}\sqrt{3+2\sqrt{2}}=\sqrt[4]{5}(\sqrt{2}+1)=\sqrt[4]{5}(\sqrt[4]{4}+1)=\sqrt[4]{20}+\sqrt[4]{5}$

3. $3\sqrt{6}+2\sqrt{12}=\sqrt{6}(3+2\sqrt{2})$ $\therefore$ $\sqrt{3\sqrt{6}+2\sqrt{12}}=\sqrt{\sqrt{6}(3+2\sqrt{2})}$
$=\sqrt[4]{6}\sqrt{3+2\sqrt{2}}=\sqrt[4]{6}(\sqrt{2}+1)=\sqrt[4]{6}(\sqrt[4]{4}+1)=\sqrt[4]{24}+\sqrt[4]{6}$

4. $\sqrt{18}-4=\sqrt{18}-\sqrt{16}=\sqrt{2}(\sqrt{9}-\sqrt{8})=\sqrt{2}(3-\sqrt{8})$
$\therefore\sqrt{\sqrt{18}-4}=\sqrt{\sqrt{2}(3-\sqrt{8})}=\sqrt[4]{2}\sqrt{(3-\sqrt{8})}=\sqrt[4]{2}(\sqrt{2}-1)$
$=\sqrt[4]{2}(\sqrt[4]{4}-1)=\sqrt[4]{8}-\sqrt[4]{2}$

Exercise XLVII.

1. $4\sqrt{-27}-2\sqrt{-12}=12\sqrt{-3}-4\sqrt{-3}=8\sqrt{-3}$
$(a+\sqrt{-b})+(a+\sqrt{-c})=2a+(\sqrt{b}+\sqrt{c})\sqrt{-1}$

2. $\sqrt{-5}+\sqrt{-7}+\sqrt{-11}=(\sqrt{5}+\sqrt{7}+\sqrt{11})\sqrt{-1}$

3. $\sqrt{7+6\sqrt{-2}}=\sqrt{x}+\sqrt{y}$; $\sqrt{7-6\sqrt{-2}}=\sqrt{x}-\sqrt{y}$

$$\sqrt{49+72}=11=x-y$$
$$\text{And } 7=x+y$$
$$\therefore x=9, \text{ and } y=-2$$

4. $(4\sqrt{-3}+7\sqrt{-2})(4\sqrt{-3}-7\sqrt{-2})=(4\sqrt{-3})^2-(7\sqrt{-2})^2$
$=(16\times-3)-(49\times-2)=-48-(-98)=-48+98=50$

5. $(\sqrt{-2}-3\sqrt{-3})^2=-2-6\sqrt{6}(\sqrt{-1})^2+(9\times-3)=-29+6\sqrt{6}$

6. $\dfrac{\sqrt{2}-\sqrt{-5}}{(\sqrt{2}+\sqrt{-5})(\sqrt{2}-\sqrt{-5})}=\dfrac{\sqrt{2}-\sqrt{-5}}{2-(-5)}=\dfrac{\sqrt{2}-\sqrt{-5}}{7}$

7. $a^{123}\times-\sqrt{-1}=-a^{123}\sqrt{-1}$; $+1$; $\sqrt{-1}$; -1 [See Algebra Art. 193 (III)]

8. $(a-\sqrt{-a})^2=a^2-2a\sqrt{-a}+(-a)=a^2-2a\sqrt{-a}-a$

9. $(\sqrt{2}-\sqrt{-4})^3 = (\sqrt{2})^3 - 3(\sqrt{2})^2(\sqrt{-4}) + 3\sqrt{2}(\sqrt{-4})^2 - (\sqrt{-4})^3$
$= \sqrt{8} - 6\sqrt{4}\sqrt{-1} + 3\sqrt{2}(-4) - (\sqrt{4}\sqrt{-1})^3 = 2\sqrt{2} - 12\sqrt{-1} - 12\sqrt{2}$
$+ 8\sqrt{-1} = -4\sqrt{-1} - 10\sqrt{2}$

10. Let $\sqrt{-2-2\sqrt{-15}} = \sqrt{x} - \sqrt{y}$; then $\sqrt{-2+2\sqrt{-15}}$
$= \sqrt{x} + \sqrt{y}$ $\therefore$ $\sqrt{4-4(-15)} = \sqrt{64} = 8 = x - y$, and $x + y = -2$
$\therefore$ $2x = 6$; $x = 3$; $2y = -10$; and $y = -5$. Hence $\sqrt{x} - \sqrt{y}$
$= \sqrt{3} - \sqrt{-5}$

11. We are to find the square root of $0 \pm \sqrt{-1}$
Let $\sqrt{0 \pm \sqrt{-1}} = \sqrt{x} \pm \sqrt{y}$
then $\sqrt{0 \mp \sqrt{-1}} = \sqrt{x} \mp \sqrt{y}$
or $\sqrt{0-(-1)} = \sqrt{1} = 1 = x - y$, and $x + g = 0$.
$\therefore$ $x = \frac{1}{2}$, and $y = -\frac{1}{2}$
Hence $\sqrt{0 \pm \sqrt{-1}}$; that is of $\pm\sqrt{-1} = \sqrt{\frac{1}{2}} \pm \sqrt{-\frac{1}{2}}$
$= \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{-2}$; or $\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{-2}$

12. Let $\sqrt{31+42\sqrt{-2}} = \sqrt{x} + \sqrt{y}$; then $\sqrt{31-42\sqrt{-2}} = \sqrt{x} - \sqrt{y}$
$\therefore$ $\sqrt{961 - 1764(-2)} = \sqrt{961 + 3528} = \sqrt{4489} = 67 = x - y$
and $31 = x + y$ $\therefore$ $2x = 98$, and $x = 49$; $2y = -36$, or $y = -18$
Hence $\sqrt{x} + \sqrt{y} = \sqrt{49} + \sqrt{-18} = 7 + 3\sqrt{-2}$

13. $\dfrac{4+\sqrt{-2}}{2-\sqrt{-2}} = \dfrac{(4+\sqrt{-2})(2+\sqrt{-2})}{(2-\sqrt{-2})(2+\sqrt{-2})} = \dfrac{8+4\sqrt{-2}+2\sqrt{-2}-2}{4-(-2)}$
$= \dfrac{6+6\sqrt{-2}}{6} = 1 + \sqrt{-2}$

14. $7-\sqrt{-5}\,)\ 14 - \sqrt{15} - 7\sqrt{-3} - 2\sqrt{-5}\,(\,2 - \sqrt{-3}$

$$\begin{array}{l} \underline{14 - 2\sqrt{-5}} \\ \quad -7\sqrt{-3} - \sqrt{15} \\ \quad -7\sqrt{-3} - \sqrt{15}^{*} \end{array}$$

15. $(a+b\sqrt{-1})(a-b\sqrt{-1}) = a^2 - (b\sqrt{-1})^2 = a^2 - (b^2 \times -1)$
$= a^2 - (-b^2) = a^2 + b^2$

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*Thus $-\sqrt{-5} \times -\sqrt{-3} = -\sqrt{5}\sqrt{-1} \times -\sqrt{3}\sqrt{-1} = \sqrt{15}(\sqrt{-1})^2$
$= \sqrt{15} \times -1 = -\sqrt{15}$

Exercise XLVIII.

1. $12 + x = 4 + 4\sqrt{x} + x$; or $4\sqrt{x} = 8$; or $\sqrt{x} = 2$; or $x = 4$

2. $3x - 6 = 2x$; or $x = 6$

3. $x - 24 = x - 4\sqrt{x} + 4$; or $4\sqrt{x} = 28$; or $\sqrt{x} = 7$; or $x = 49$

4. $x - 2\sqrt{x}\sqrt{\dfrac{a}{x}} + ax^{-1} = a + x$, or $-2\sqrt{a} + ax^{-1} = a$;

$-2 + \dfrac{\sqrt{a}}{x} = \sqrt{a}$; or $\dfrac{\sqrt{a}}{x} = \sqrt{a} + 2$; or $\dfrac{x}{\sqrt{a}} = \dfrac{1}{\sqrt{a} + 2}$; or $x = \dfrac{\sqrt{a}}{2 + \sqrt{a}}$

5. $\sqrt{\sqrt{\sqrt{\sqrt{x + 123} + 4} + 5} + 6} + 7 = 4$; or

$\sqrt{\sqrt{\sqrt{\sqrt{x + 123} + 4} + 5} + 6} = -3$; or $\sqrt{\sqrt{\sqrt{x + 123} + 4} + 5} + 6 = 9$;

or $\sqrt{\sqrt{\sqrt{x + 123} + 4} + 5} = 3$; or $\sqrt{\sqrt{x + 123} + 4} + 5 = 9$; or

$\sqrt{\sqrt{x + 123} + 4} = 4$; or $\sqrt{x + 123} + 4 = 16$; or $\sqrt{x + 123} = 12$; or

$x + 123 = 144$; or $x = 21$

6. $\sqrt{ax} - \sqrt{x} = \sqrt{a}$; or $\sqrt{x}(\sqrt{a} - 1) = \sqrt{a}$; or $\sqrt{x} = \dfrac{\sqrt{a}}{\sqrt{a} - 1}$;

or $x = \dfrac{a}{(\sqrt{a} - 1)^2}$

7. $2x + \sqrt{x^4 - x^2} = x^2 + 2x + 1$; or $\sqrt{x^4 - x^2} = x^2 + 1$; or $x^4 - x^2$
$= x^4 + 2x^2 + 1$; or $3x^2 = -1$; or $x^2 = -\frac{1}{3}$; or $x = \pm\sqrt{-\frac{1}{3}} = \pm\frac{1}{3}\sqrt{-3}$

8. $x + 34\sqrt{x} + 168 = 152 + 42\sqrt{x} + x$; or $8\sqrt{x} = 16$; or $\sqrt{x} = 2$;
or $x = 4$

9. $\sqrt{x} + \sqrt{x + 2} = \dfrac{4}{\sqrt{x + 2}}$; or $\sqrt{x}\sqrt{x + 2} + x + 2 = 4$; or $\sqrt{x}\sqrt{x + 2}$
$= 2 - x$; or $x(x + 2) = 4 - 4x + x^2$; or $x^2 + 2x = 4 - 4x + x^2$; or
$6x = 4$; or $x = \frac{2}{3}$

10. $a + x + 2\sqrt{a^2 - x^2} + a - x = ax$; or $2\sqrt{a^2 - x^2} = ax - 2a$; or
$4a^2 - 4x^2 = a^2x^2 - 4a^2x + 4a^2$; or $a^2x^2 + 4x^2 = 4a^2x$; or $a^2x + 4x$
$= 4a^2$; or $x(a^2 + 4) = 4a^2$; or $x = \dfrac{4a^2}{a^2 + 4}$

11. $a^2 + 2ax + x^2 = a^2 + x\sqrt{b^2 + x^2}$; or $2ax + x^2 = x\sqrt{b^2 + x^2}$; or $2a + x = \sqrt{b^2 + x^2}$; or $4a^2 + 4ax + x^2 = b^2 + x^2$; or $4ax = b^2 - 4a^2$; or $x = \dfrac{b^2 - 4a^2}{4a}$

12. $\sqrt{b^2 + ax + x^2} = a - b - x$; or $b^2 + ax + x^2 = a^2 - 2ab - 2ax + b^2 + 2bx + x^2$; or $3ax - 2bx = a^2 - 2ab$; or $x(3a - 2b) = a^2 - 2ab$; or $x = \dfrac{a^2 - 2ab}{3a - 2b}$.

13. $x + 2a\sqrt{x} + 3b\sqrt{x} + 6ab = 4ab + b\sqrt{x} + 4a\sqrt{x} + x$; or $2b\sqrt{x} - 2a\sqrt{x} = - 2ab$; or $\sqrt{x} = \dfrac{ab}{a - b}$; or $x = \dfrac{a^2b^2}{(a - b)^2}$

14. $\sqrt{x} + \sqrt{4a + x} = \dfrac{2a}{\sqrt{1 + x}}$; or $\sqrt{x + x^2} + \sqrt{4a + 4ax + x + x^2} = 2a$; or $\sqrt{4a + 4ax + x + x^2} = 2a - \sqrt{x + x^2}$; or $4a + 4ax + x + x^2 = 4a^2 - 4a\sqrt{x + x^2} + x + x^2$; or $4a + 4ax - 4a^2 = - 4a\sqrt{x + x^2}$; or $1 + x - a = -\sqrt{x^2 + x}$; or $1 + 2x - 2a + x^2 - 2ax + a^2 = x^2 + x$; or $x - 2ax = 2a - a^2 - 1$; or $x = \dfrac{2a - a^2 - 1}{1 - 2a}$

15. $x - 32 = 256 - 32\sqrt{x} + x$; or $32\sqrt{x} = 288$; or $\sqrt{x} = 9$; or $x = 81$.

16. $\dfrac{b}{a + x} + 2\left(\dfrac{bc}{a^2 - x^2}\right)^{\frac{1}{2}} + \dfrac{c}{a - x} = \left(\dfrac{4bc}{a^2 - x^2}\right)^{\frac{1}{2}}$; or $\dfrac{b}{a + x} + \dfrac{c}{a - x} = 0$; or $ab - bx + ac + cx = 0$; or $bx - cx = ab + ac$; or $x = \dfrac{a(b + c)}{b - c}$

17. $x + \sqrt{x} - \sqrt{x^2 - x} = \dfrac{3\sqrt{x}}{2}$ = given equation multiplied by $\sqrt{x + \sqrt{x}}$; or $\sqrt{x + 1} - \sqrt{x - 1} = \frac{3}{2}$ (dividing by $\sqrt{x}$); or $2\sqrt{x + 2} = 2\sqrt{x - 1} + 3$; or $2\sqrt{x} - 1 = 2\sqrt{x - 1}$; or $4x - 4\sqrt{x} + 1 = 4x - 4$; or $4\sqrt{x} = 5$; or $16x = 25$; or $x = \frac{25}{16}$.

18. $x + a = c^2 - 2c\sqrt{x + b} + x + b$; or $- 2c\sqrt{x + b} = a - b - c^2$; or $4c^2(x + b) = (a - b - c^2)^2$; or $4c^2x = (a - c^2 - b)^2 - 4c^2b$; or $x = \dfrac{(a - c^2 - b)^2 - 4c^2b}{4c^2} = \left(\dfrac{a - c^2 - b}{2c}\right)^2 - b$

19. $\frac{1}{x^2}+\frac{2}{ax}+\frac{1}{a^2}=\frac{1}{a^2}+\sqrt{\frac{4}{a^2x^2}+\frac{9}{x^4}}$; or $\frac{1}{x}+\frac{2}{a}=\sqrt{\frac{4}{a^2}+\frac{9}{x^2}}$; or $\frac{1}{x^2}+\frac{4}{ax}+\frac{4}{a^2}=\frac{4}{a^2}+\frac{9}{x^2}$; or $\frac{1}{x}+\frac{4}{a}=\frac{9}{x}$; or $a+4x=9a$; or $4x=8a$; or $x=2a$

20. $\frac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}}=\frac{m}{1}$ or $\frac{2\sqrt{x+a}}{2\sqrt{x-a}}=\frac{m+1}{m-1}$;

or $\frac{\sqrt{x+a}}{\sqrt{x-a}}=\frac{m+1}{m-1}$; or $\frac{x+a}{x-a}=\frac{m^2+2m+1}{m^2-2m+1}$;

or $\frac{2x}{2a}=\frac{2(m^2+1)}{4m}$; or $\frac{x}{a}=\frac{m^2+1}{2m}$; or $x=\frac{a(m^2+1)}{2m}$

Exercise XLIX.

1. $x^2=9$; or $x=\pm 3$

2. $18-18x+18+18x=100-100x^2$; or $100x^2=64$; or $x^2=\frac{64}{100}$; or $x=\pm\frac{4}{5}$

3. $4x^2=3x^2+9$; or $x^2=9$; or $x=\pm 3$

4. $4x^2-8=1$; or $4x^2=9$; or $x^2=\frac{9}{4}$; or $x=\pm\frac{3}{2}$

5. $x^2-6x+9=13-6x$; or $x^2=4$; or $x=\pm 2$

6. $3(x^2+10x+25)-7x=23x$; or $3x^2=-75$; or $x^2=-25$; or $x=\pm 5\sqrt{-1}$

7. $10x^2+17-10x^2+8=\frac{216x^2+36}{11x^2-8}$; or $25=\frac{216x^2+36}{11x^2-8}$; or $275x^2-200=216x^2+36$; or $59x^2=236$; or $x^2=4$; or $x=\pm 2$

8. $\sqrt{9+2x^2}=9$; or $9+2x^2=81$; or $2x^2=72$; or $x^2=36$; or $x=\pm 6$

9. $\sqrt{(x-3)(x+3)}=3a$; or $x^2-9=9a^2$; or $x^2=9a^2+9$; or $x=\pm 3\sqrt{a^2+1}$

10. $\frac{\sqrt{a^2 - x^2}}{x} = \frac{x}{b} - \frac{a}{x}$; or $\frac{a^2 - x^2}{x^2} = \frac{x^2}{b^2} - \frac{2a}{b} + \frac{a^2}{x^2}$; or $\frac{a^2}{x^2} - 1 = \frac{x^2}{b^2} - \frac{2a}{b} + \frac{a^2}{x^2}$; or $\frac{x^2}{b^2} = \frac{2a}{b} - 1$; or $x^2 = 2ab - b^2$; or $x = \pm\sqrt{2ab - b^2}$

11. $\frac{a^2}{x^2} + b^2 - 2\sqrt{\frac{a^4}{x^4} - b^4} + \frac{a^2}{x^2} - b^2 = b^2$; or $\frac{2a^2}{x^2} - b^2 = 2\sqrt{\frac{a^4}{x^4} - b^4}$ or $\frac{4a^4}{x^4} - \frac{4a^2b^2}{x^2} + b^4 = \frac{4a^4}{x^4} - 4b^4$; or $\frac{4a^2b^2}{x^2} = 5b^4$; or $\frac{4a^2}{x^2} = 5b^2$; or $x^2 = \frac{4a^2}{5b^2}$ or $x = +\frac{2a}{b}\sqrt{\tfrac{1}{5}} = \pm\frac{2a}{5b}\sqrt{5}$

12. $3ax^2 - cx^2 = d - 1 - b$; or $x^2(3a - c) = d - 1 - b$; or $x^2 = \frac{d - 1 - b}{3a - c}$; or $x = \pm\sqrt{\left(\frac{d - 1 - b}{3a - c}\right)}$

13. $\sqrt{a^2 - x^2} = a^2\sqrt{1 - x^2} - x\sqrt{a^2 - 1}$;
$a^2 - x^2 = a^4 - a^4x^2 - 2a^2x\sqrt{(1 - x^2)(a^2 - 1)} + a^2x^2 - x^2$;
or $2x\sqrt{(1 - x^2)(a^2 - 1)} = a^2 - a^2x^2 - 1 + x^2$; or $2x\sqrt{(1 - x^2)(a^2 - 1)} = (a^2 - 1)(1 - x^2)$; or $2x = \sqrt{(a^2 - 1)(1 - x^2)}$; or $4x^2 = (a^2 - 1)(1 - x^2) = a^2 - a^2x^2 + x^2 - 1$; or $3x^2 + a^2x^2 = a^2 - 1$; or $x^2 = \frac{a^2 - 1}{3 + a^2}$; or $x = \pm\left(\frac{a^2 - 1}{3 + a^2}\right)^{\frac{1}{2}}$

14. $x\sqrt{b^2 + x^2} + b^2 + x^2 = cb^2$; $x\sqrt{b^2 + x^2} = cb^2 - b^2 - x^2$; or $b^2x^2 + x^4 = c^2b^4 - 2cb^4 - 2cb^2x^2 + b^4 + 2b^2x^2 + x^4$; or $2cb^2x^2 - b^2x^2 = c^2b^4 - 2cb^4 + b^4$; or $2cx^2 - x^2 = c^2b^2 - 2cb^2 + b^2 = (c^2 - 2c + 1)b^2$; or $(2c - 1)x^2 = (c - 1)^2b^2$; or $x^2 = \frac{(c - 1)^2b^2}{2c - 1}$; or $x = \pm\frac{(c - 1)b}{\sqrt{2c - 1}}$

15. $3 + \tfrac{1}{4}x - 2\sqrt{\frac{2x}{3}\sqrt{3 + \tfrac{1}{4}x}} + \tfrac{3}{4}x = \tfrac{1}{4}x - 3$; or $-2\sqrt{\frac{2x}{3}\sqrt{3 + \tfrac{1}{4}x}} = -6 - \frac{2x}{3}$; or $\sqrt{\frac{2x}{3}\sqrt{3 + \tfrac{1}{4}x}} = 3 + \frac{x}{3}$; or $\frac{2x}{3}(3 + \tfrac{1}{4}x) = 9 + 2x + \frac{x^2}{9}$ or $\frac{x^2}{6} = 9 + \frac{x^2}{9}$; or $\frac{x^2}{18} = 9$; or $x^2 = 81 \times 2$; or $x = \pm 9\sqrt{2}$

16. $a + x + 3(a + x)^{\frac{2}{3}}(a - x)^{\frac{1}{3}} + 3(a + x)^{\frac{1}{3}}(a - x)^{\frac{2}{3}} + a - x = b^3$; or $2a + 3(a^2 - x^2)^{\frac{1}{3}}(\sqrt[3]{a + x} + \sqrt[3]{a - x}) = b^3$; and by given equation $\sqrt[3]{a + x} + \sqrt[3]{a - x} = b$; substituting this in the last equation we nave $2a + 3b\sqrt[3]{a^2 - x^2} = b^3$; or $3b\sqrt[3]{a^2 - x^2} = b^3 - 2a$; or $\sqrt[3]{a^2 - x^2} = \dfrac{b^3 - 2a}{3b}$; or $a^2 - x^2 = \left(\dfrac{b^3 - 2a}{3b}\right)^3$; or $x^2 = a^2 - \left(\dfrac{b^3 - 2a}{3b}\right)^3$

$$\therefore x = \sqrt{a^2 - \left(\frac{b^3 - 2a}{3b}\right)^3}$$

Exercise L

1. $2x^2 + 8x = 90$; or $x^2 + 4x = 45$; or $x^2 + 4x + 4 = 49$; or $x + 2 = \pm 7$; $x = 5$; or $- 9$.

2. $x^2 - 19 = 8x - 10$; or $x^2 - 8x = 9$; or $x^2 - 8x + 16 = 25$; or or $x - 4 = \pm 5$; $x = 9$; or $- 1$

3. $x^2 - 8x = 20$; or $x^2 - 8x + 16 = 36$; or $x - 4 = \pm 6$; $x = 10$; or $- 2$

4. $x^2 + 12x = 45$; or $x^2 + 12x + 36 = 81$; or $x + 6 = \pm 9$; $x = 3$; or $- 15$

5. $3x^2 + 2x = 85$; or $x^2 + \frac{2}{3}x = \frac{85}{3}$; or $x^2 + \frac{2}{3}x + \frac{1}{9} = \frac{255}{9} + \frac{1}{9} = \frac{256}{9}$; or $x + \frac{1}{3} = \pm \frac{16}{3}$; $x = 5$; or $- 5\frac{2}{3}$

6. $3x^2 - 14x = - 15$; or $x^2 - \frac{14}{3}x = - 5$; or $x^2 - \frac{14}{3}x + \frac{49}{9} = - 5 + \frac{49}{9} = \frac{4}{9}$; or $x - \frac{7}{3} = \pm \frac{2}{3}$; or $x = 3$; or $1\frac{2}{3}$

7. $5x^2 - 236x = - 47$; or $x^2 - \frac{236}{5}x = - \frac{47}{5}$; or $x^2 - \frac{236}{5}x + \frac{13924}{25} = - \frac{47}{5} + \frac{13924}{25} = \frac{13689}{25}$; or $x - \frac{117}{5} = \pm \frac{117}{5}$; or $x = \frac{235}{5}$ or $\frac{1}{5}$; that is $x = 47$ or $\frac{1}{5}$

8. $4x^2 - 8x = 320$; or $x^2 - 2x = 80$; $x^2 - 2x + 1 = 81$; $x - 1 = \pm 9$; $x = 10$ or $- 8$

9. $x^2 - 2x = - a^2$; or $x^2 - 2x + 1 = 1 - a^2$; or $x - 1 = \sqrt{1 - a^2}$; or $x = 1 \pm \sqrt{1 - a^2}$

10. $5x^2 + 4x = 273$; $x^2 + \frac{4}{5}x = \frac{273}{5}$; $x^2 + \frac{4}{5}x + \frac{4}{25} = \frac{1365}{25} + \frac{4}{25}$ $= \frac{1369}{25}$; $x + \frac{2}{5} = \pm\frac{37}{5}$; $x = 7$ or $-7\frac{4}{5}$

11. $7x^2 - 20x = 32$; $x^2 - \frac{20}{7}x = \frac{32}{7}$; $x^2 - \frac{20}{7}x + \frac{100}{49} = \frac{224}{49} + \frac{100}{49}$ $= \frac{324}{49}$; $x - \frac{10}{7} = \pm\frac{18}{7}$; $x = 4$ or $-1\frac{1}{7}$

12. $x^2 - 7x = -12$; $x^2 - 7x + \frac{49}{4} = -\frac{48}{4} + \frac{49}{4} = \frac{1}{4}$; $x - \frac{7}{2} = \pm\frac{1}{2}$; $x = 4$ or 3

13. $3x^2 - 11x = -6$; or $x^2 - \frac{11}{3}x = -2$; or $x^2 - \frac{11}{3}x + \frac{121}{36}$ $= -2 + \frac{121}{36} = \frac{49}{36}$; $x - \frac{11}{6} = \pm\frac{7}{6}$; $x = 3$ or $\frac{2}{3}$

14. $acx^2 + bcx - adx = bd$; or $x^2 + \dfrac{bc - ad}{ac}x = \dfrac{bd}{ac}$;

or $x^2 + \dfrac{bc - ad}{ac}x + \left(\dfrac{bc - ad}{2ac}\right)^2 = \dfrac{bd}{ac} + \dfrac{(bc - ad)^2}{4a^2c^2}$;

or $x^2 + \dfrac{bc - ad}{ac} + \left(\dfrac{bc - ad}{2ac}\right)^2 = \dfrac{4abcd + b^2c^2 - 2abcd + a^2d^2}{4a^2c^2}$

$= \dfrac{b^2c^2 + 2abcd + a^2d^2}{4a^2c^2} = \left(\dfrac{bc + ad}{2ac}\right)^2$; or $x + \dfrac{bc - ad}{2ac} = \pm\dfrac{bc + ad}{2ac}$;

or $x = \pm\dfrac{bc + ad}{2ac} - \dfrac{bc - ad}{2ac}$; $x = \dfrac{2ad}{2ac} = \dfrac{d}{c}$; or $x = -\dfrac{2bc}{2ac} = -\dfrac{b}{a}$

15. $\dfrac{1}{x - \sqrt{x}} = \dfrac{x - \sqrt{x}}{4}$ by dividing the given equation by $x + \sqrt{x}$; or $4 = x^2 - 2x\sqrt{x} + x$; or $4 = x(x - 2\sqrt{x} + 1)$; or $\dfrac{4}{x} = (\sqrt{x} - 1)^2$; or $\dfrac{2}{\sqrt{x}} = \pm(\sqrt{x} - 1)$ (I); or $2 = x - \sqrt{x}$; or $x - \sqrt{x} + \frac{1}{4} = \frac{8}{4} + \frac{1}{4} = \frac{9}{4}$; or $\sqrt{x} - \frac{1}{2} = \pm\frac{3}{2}$; or $\sqrt{x} = 2$ or -1 $\therefore x = 4$ or 1

Taking the minus sign in (I) we have $\dfrac{2}{\sqrt{x}} = -\sqrt{x} + 1$; or $2 = -x + \sqrt{x}$; or $x - \sqrt{x} = -2$; or $x - \sqrt{x} + \frac{1}{4} = -\frac{8}{4} + \frac{1}{4} = -\frac{7}{4}$; or $\sqrt{x} - \frac{1}{2} = \pm\sqrt{-\frac{7}{4}}$; or $\sqrt{x} = \frac{1}{2} \pm \frac{1}{2}\sqrt{-7} = \frac{1}{2}(1 \pm \sqrt{-7})$; or $x = \frac{1}{4}(1 \pm 2\sqrt{-7} - 7) = \frac{1}{4}(-6 \pm 2\sqrt{-7}) = \frac{1}{2}(-3 \pm \sqrt{-7})$

The rejected factor $x + \sqrt{x} = 0$ gives us $x = 0$ or 1

16. $x^2 - x = 210$; $x^2 - x + \frac{1}{4} = \frac{840}{4} + \frac{1}{4} = \frac{841}{4}$; $x - \frac{1}{2} = \pm\frac{29}{2}$; $x = 15$ or -14.

17. $4x^2 + 36 = 3x^2 + 48 - 11x$; or $x^2 + 11x = 12$; or $x^2 + 11x + \frac{121}{4} = \frac{48}{4} + \frac{121}{4} = \frac{169}{4}$; $x + \frac{11}{2} = \pm\frac{13}{2}$; $x = 1$ or -12

18. $\frac{x-2}{x+2} + \frac{x+2}{x-2} = \frac{x-3}{x+3} + \frac{x+4}{x-4}$; $\frac{x^2 - 4x + 4 + x^2 + 4x + 4}{x^2 - 4}$

$= \frac{x^2 - 7x + 12 + x^2 + 7x + 12}{x^2 - x - 12}$; or $\frac{2x^2 + 8}{x^2 - 4} = \frac{2x^2 + 24}{x^2 - x - 12}$;

or $\frac{x^2 + 4}{x^2 - 4} = \frac{x^2 + 12}{x^2 - x - 12}$; or $\frac{2x^2}{8} = \frac{2x^2 - x}{x + 24}$; or $\frac{x}{4} = \frac{2x - 1}{x + 24}$;

or $x^2 + 24x = 8x - 4$; or $x^2 + 16x = -4$; or $x^2 + 16x + 64 = 60$;

or $x + 8 = \sqrt{60} = \pm 2\sqrt{15}$; or $x = \pm 2\sqrt{15} - 8$

The rejected factor $x = 0$ gives us the other value.

19. $49x^2 + 42x + 9 = 10(2x^2 + 4x - 6 - 2x^2 + 3x + 9)$ $= 10(7x + 3)$; or $49x^2 + 42x + 9 = 70x + 30$; or $7x^2 - 4x = 3$; $x^2 - \frac{4}{7}$; $x = \frac{3}{7}$ $x^2 - \frac{4}{7}x + \frac{4}{49} = \frac{21}{49} + \frac{4}{49} = \frac{25}{49}$; $x - \frac{2}{7} = \pm\frac{5}{7}$; $x = 1$ or $-\frac{3}{7}$

20. $ax^2 - fx^2 - bx - cx = -b - c$; or $fx^2 - ax^2 + bx + cx = b + c$

$(f - a)x^2 + (b + c)x = b + c$; $x^2 + \left(\frac{b+c}{f-a}\right)x = \frac{b+c}{f-a}$;

$$x^2 + \left(\frac{b+c}{f-a}\right)x + \left(\frac{b+c}{2f-2a}\right)^2 = \frac{b+c}{f-a} + \left(\frac{b+c}{2f-2a}\right)^2$$

$$x = \pm\sqrt{\frac{b+c}{f-a} + \left(\frac{b+c}{2f-2a}\right)^2} - \frac{b+c}{2f-2a}$$

21. $\frac{1}{a - m + x} = \frac{1}{a} - \frac{1}{m} + \frac{1}{x}$; $\frac{1}{a - m + x} = \frac{mx - ax + am}{amx}$

$amx = 3amx - a^2x + a^2m - m^2x - am^2 + mx^2 - ax^2$; $ax^2 - mx^2 + a^2x - 2amx + m^2x = a^2m - am^2 = am(a - m)$; $(a - m)x^2 + (a^2 - 2am + m^2)x = am(a - m)$; $x^2 + (a - m)x = am$;

$x^2 + (a - m)x + \frac{(a-m)^2}{4} = am + \frac{(a-m)^2}{4} = \frac{4am + (a-m)^2}{4}$

$= \frac{a^2 + 2am + m^2}{4} = \frac{(a+m)^2}{4}$; $x + \frac{a - m}{2} = \pm\frac{a + m}{2}$;

$x = +\frac{a+m}{2} - \frac{a-m}{2} = m$ or $-a$

22. $abx^2 - 2x(a+b)\sqrt{ab} = (a-b)^2$; $x^2 - \dfrac{2(a+b)}{\sqrt{ab}}x = \dfrac{(a-b)^2}{ab}$

$x^2 - \dfrac{2(a+b)}{\sqrt{ab}} + \dfrac{(a+b)^2}{ab} = \dfrac{(a-b)^2}{ab} + \dfrac{(a+b)^2}{ab} = \dfrac{2(a^2+b^2)}{ab}$

$x - \dfrac{a+b}{\sqrt{ab}} = \pm\dfrac{\sqrt{2(a^2+b^2)}}{\sqrt{ab}}$; $x = \dfrac{a+b \pm \sqrt{2(a^2+b^2)}}{\sqrt{ab}}$

Exercise LI.

1. $3x^2 + 2x = 85$; $36x^2 + 24x = 1020$; $36x^2 + 24x + 4 = 1024$; $6x + 2 = \pm 32$; $6x = 30$ or -34; $x = 5$ or $-5\frac{2}{3}$

2. $4x^2 - 4x = 840$; $4x^2 - 4x + 1 = 841$; $2x - 1 = \pm 29$; $2x = 30$ or -28; $x = 15$ or -14

3. $64x^2 - 48x = 1360$; $64x^2 - 48x + 9 = 1369$; $8x - 3 = \pm 37$; $x = 5$ or $-4\frac{1}{4}$

4. $x^2 - 26x = -25$; $4x^2 - 104x + 676 = -100 + 676 = 576$; $2x - 26 = \pm 24$; $2x = 50$ or 2; $x = 25$ or 1

5. $5x^2 + 4x = 273$; $100x^2 + 80x = 5460$; $100x^2 + 80x + 16 = 5476$; $10x + 4 = \pm 74$; $10x = 70$ or -78; $x = 7$ or $-7\frac{4}{5}$

6. $4x^2 + 8x = 21$; $x^2 + 2x = \frac{21}{4}$; $4x^2 + 8x + 4 = 21 + 4 = 25$; $2x + 2 = \pm 5$; $2x = 3$ or -7; $x = 1\frac{1}{2}$ or $-3\frac{1}{2}$

7. $11x^2 + 7x - 4 = 14x$; $11x^2 - 7x = 4$; $484x^2 - 308x = 176$; $484x^2 - 308x + 49 = 176 + 49 = 225$; $22x - 7 = \pm 15$; $x = 1$ or $-\frac{4}{11}$

8. $a^2x^2 + (ab - ac)x = bc$; $ax^2 + (b-c)x = \dfrac{bc}{a}$; $4a^2x^2 + 4a(b-c)x + (b-c)^2 - 4bc + (b-c)^2 = (b+c)^2$; $2ax + b - c = \pm(b+c)$; $2ax = 2c$ or $-2b$; $x = \dfrac{c}{a}$ or $-\dfrac{b}{a}$

9. $12x^2 + 120 = 16x + 135$; $12x^2 - 16x = 15$; $3x^2 - 4x = \frac{15}{4}$; $36x^2 - 48x = 45$; $36x^2 - 48x + 16 = 45 + 16 = 61$; $6x - 4 = \pm\sqrt{61}$; $6x = 4 \pm \sqrt{61}$, $x = \frac{1}{6}(4 \pm \sqrt{61})$

10. $7x^2 - 4x^2\sqrt{3} + (2 - \sqrt{3})x = 2$; $(7 - 4\sqrt{3})x^2 + (2 - \sqrt{3})x = 2$;

$x^2 + \frac{x}{2 - \sqrt{3}} = \frac{2}{7 - 4\sqrt{3}}$ [since $7 - 4\sqrt{3} = (2 - \sqrt{3})^2$]

$$4x^2 + \frac{4}{2 - \sqrt{3}}x + \frac{1}{7 - 4\sqrt{3}} = \frac{8}{7 - 4\sqrt{3}} + \frac{1}{7 - 4\sqrt{3}} = \frac{9}{7 - 4\sqrt{3}}$$

$$2x + \frac{1}{2 - \sqrt{3}} = \pm\frac{3}{2 - \sqrt{3}};\ 2x = -\frac{4}{2 - \sqrt{3}} \text{ or } \frac{2}{2 - \sqrt{3}}$$

$$x = \frac{1}{2 - \sqrt{3}} \text{ or } -\frac{2}{2 - \sqrt{3}}$$

11. $x^2 + 6ax = b^2$; $x^2 + 6ax + 9a^2 = b^2 + 9a^2$; $x + 3a = \pm\sqrt{9a^2 + b^2}$; $x = \pm\sqrt{9a^2 + b^2} - 3a$

12. $\frac{45 - 9x}{x + 3} - 3x = x - \frac{63 + 36x}{19}$; $\frac{19(45 - 9x)}{3 + x} = 40x - 63$;

$855 - 171x = 40x^2 + 57x - 189$; $10x^2 + 57x = 261$;

$400x^2 + 2280x + (57)^2 = 10440 + 3249 = 13689$;

$20x + 57 = \pm 117$; $20x = 60$ or -174; $x = 3$ or $-8\frac{7}{10}$

13. $x^2 - 5x = -m^2$; $4x^2 - 20x + 25 = -4m^2 + 25$; $2x - 5 = \pm\sqrt{25 - 4m^2}$; $2x = 5 \pm \sqrt{25 - 4m^2}$; $x = \frac{1}{2}(5 \pm \sqrt{25 - 4m^2})$

14. $mx^2 - nx^2 - 2mx\sqrt{n} = -mn$; $(m - n)x^2 - (2m\sqrt{n})x = -mn$

$$x^2 - \frac{2m\sqrt{n}}{m - n}x = -\frac{mn}{m - n};\ 4x^2 - \frac{8m\sqrt{n}}{m - n}x = -\frac{4mn}{m - n}$$

$$4x^2 - \frac{8m\sqrt{n}}{m - n} + \frac{4m^2n}{(m - n)^2} = \frac{4m^2n}{(m - n)^2} - \frac{4mn}{m - n} = \frac{4mn^2}{(m - n)^2}$$

$$2x - \frac{2m\sqrt{n}}{m - n} = \pm\frac{2n\sqrt{m}}{m - n};\ 2x = \frac{2n\sqrt{m} + 2m\sqrt{n}}{m - n} \text{ or } \frac{2m\sqrt{n} - 2n\sqrt{m}}{m - n}$$

$$x = \frac{\sqrt{mn}(\sqrt{n} + \sqrt{m})}{m - n} \text{ or } \frac{\sqrt{mn}(\sqrt{m} - \sqrt{n})}{m - n};\ x = \frac{\sqrt{mn}}{\sqrt{m} - \sqrt{n}} \text{ or } \frac{\sqrt{mn}}{\sqrt{m} + \sqrt{n}}$$

15. $1+x+x^2=\left(\dfrac{a+1}{a-1}\right)\left(\dfrac{1+x^2+x^4}{1+x+x^2}\right)=\left(\dfrac{a+1}{a-1}\right)\left(\dfrac{1+x^2+x^4+x-x}{1+x+x^2}\right)$

$1+x+x^2=\left(\dfrac{a+1}{a-1}\right)\left(\dfrac{1+x+x^2-x(1-x^3)}{1+x+x^2}\right)$

$=\left(\dfrac{a+1}{a-1}\right)\left(\dfrac{1+x+x^2-x(1-x)(1+x+x^2)}{1+x+x^2}\right)$

$1+x+x^2=\left(\dfrac{a+1}{a-1}\right)\left\{\dfrac{(1+x+x^2)(1-x+x^2)}{1+x+x^2}\right\}=\left(\dfrac{a+1}{a-1}\right)(1-x+x^2)$

$\dfrac{1+x+x^2}{1-x+x^2}=\dfrac{a+1}{a-1}$; $\dfrac{2+2x^2}{2x}=\dfrac{2a}{2}$; $\dfrac{1+x^2}{x}=a$; $1+x^2=ax$; $x^2-ax=-1$; $4x^2-4ax+a^2=a^2-4$, $2x-a=\pm\sqrt{a^2-4}$; $x=\frac{1}{2}(a\pm\sqrt{a^2-4})$

16. $x^4+3x^3+6=x^4+3x^3+13x^2+7x-60$; $13x^2+7x=66$; $676x^2+364x=3432$; $676x^2+364x+49=3481$; $26x+7=\pm 59$; $26x=52$ or -66; $x=2$ or $-2\frac{7}{13}$

Exercise LII.

1. $x+2=0$, and $x+7=0$ $\therefore$ $(x+2)(x+7)=x^2+9x+14=0$

2. $(x-4)(x+2)(x-1)x=0$; or $x^4-3x^3-6x^2+8x=0$

3. $(x-2)(x+2)(x-3)(x+3)x=0$; or $(x^2-4)(x^2-9)x=0$; or $x^5-13x^3+36x=0$

4. $(x-5)(x+5)(x-2)(x+2)(x-3-\sqrt{2})(x-3+\sqrt{2})=0$; or $(x^2-25)(x^2-4)\{(x-3)^2-(\sqrt{2})^2\}=(x^2-25)(x^2-4)(x^2-6x+7)=0$; or $x^6-6x^5-22x^4+174x^3-103x^2-600x+700=0$

5. $(x-1)(x-2)(x-3)(x-4)(x-5-\sqrt{6})(x-5+\sqrt{6})=0$; or $(x^2-3x+2)(x^2-7x+12)(x^2-10x+19)=0$; $x^6-20x^5+154x^4-590x^3+1189x^2-1190x+456=0$

6. $(x-5)(x-4)(x-1)x\,(x-2-\sqrt{-3})(x-2+\sqrt{-3})=0$; or $(x^2-9x+20)(x^2-x)(x^2-4x+7)=0$; $x^6-14x^5+76x^4-206x^3+283x^2-140x=0$

7. $(x-5)(x+2)=0$; or $x^2-3x-10=0$, and $(x^4-6x^3+5x^2+12x-60)\div(x^2-3x-10)$ gives us $x^2-3x+6=0$; then $x^2-3x=-6$; $x^2-3x+\frac{9}{4}=\frac{9}{4}-\frac{24}{4}=-\frac{15}{4}$; $x-\frac{3}{2}=\pm\frac{1}{2}\sqrt{-15}$; $x=\frac{1}{2}(3\pm\sqrt{-15})$

8. $(x-1-\sqrt{-6})(x-1+\sqrt{-6})=0$; or $x^2-2x+7=0$; and therefore $(x^4-4x^3+8x^2-8x-21)\div(x^2-2x+7)$ gives us $x^2-2x-3=0$. Then $x^2-2x=3$; $x^2-2x+1=4$; $x-1=\pm 2$; $x=3$ or -1

9. $(x^3+6x^2-3920)\div(x-14)$ gives us $x^2+20x+280=0$; $x^2+20x=-280$; $x^2+20x+100=-180$; $x+10=\pm 6\sqrt{-5}$; $x=-10\pm 6\sqrt{-5}$

10. $x=0$ is evidently another root, then $(x^4-6x^3+13x^2-10x)\div(x^2-2x)$, gives us $x^2-4x+5=0$; $x^2-4x=-5$; $x^2-4x+4=-1$; $x-2=\pm\sqrt{-1}$; $x=2\pm\sqrt{-1}$

11. $(x-3)(x+4)x=0$; therefore $(x^5-2x^4-25x^3+26x^2+120x)\div(x^3+x^2-12x)$ gives us $x^2-3x-10=0$; $x^2-3x=10$; $4x^2-12x+9=49$; $2x-3=\pm 7$; $2x=10$ or -4; $x=5$ or -2

12. $x=0$ is obviously another root. Then $(x-\sqrt{-2})(x+\sqrt{-2})=x^2+2=0$

$\therefore (x^4-x^3-2x-4)\div(x^2+2)=x^2-x-2=0$; that is $x^2-x=2$, whence $x=2$ or -1

13. Alg. Art. 206, when the roots are equal $4^2=4\times 2\times c$; or $16=8c$; or $c=2$

14. Alg. Art. 208 (Cor.), $\beta+\gamma=-\frac{b}{a}$ and $\beta\gamma=\frac{c}{a}$ $\therefore$ $\frac{\beta+\gamma}{\beta\gamma}=-\frac{b}{a}\div\frac{c}{a}=-\frac{b}{c}$ $\therefore$ $\frac{1}{\beta}+\frac{1}{\gamma}=-\frac{b}{c}$; Also since $\beta\gamma=\frac{c}{a}$, $\frac{1}{\beta\gamma}=\frac{a}{c}$

Hence $\frac{b}{c}$ = the sum of the roots and $\frac{a}{c}$ = their product and the equation is $x^2+\frac{b}{c}x+\frac{a}{c}=0$ that is $cx^2+bx+a=0$

15. Alg. Art. 208 $\beta + \gamma = -p$ and $\beta\gamma = q$

$\beta^2 + \gamma^2 = \beta^2 + \gamma^2 + 2\beta\gamma - 2\beta\gamma = (\beta^2 + 2\beta\gamma + \gamma^2) - 2\beta\gamma = (\beta+\gamma)^2 - 2\beta\gamma = p^2 - 2q$ (I)

$(\beta - \gamma)^2 = \beta^2 - 2\beta\gamma + \gamma^2 = (\beta^2 + \gamma^2 - 2\beta\gamma) = p^2 - 2q - 2q$ from (I) $\therefore = p^2 - 4q$ (II)

$\beta^2 - \gamma^2 = (\beta + \gamma)(\beta - \gamma) = -p(\pm\sqrt{p^2 - 4q})$, since from (II) $\beta - \gamma = \pm\sqrt{p^2 - 4q})$;

$\dfrac{1}{\beta} + \dfrac{1}{\gamma} = \dfrac{\beta + \gamma}{\beta\gamma} = -\dfrac{p}{q}$

$\beta^3 - \gamma^3 = (\beta^2 + \beta\gamma + \gamma^2)(\beta - \gamma) = (\beta^2 + \gamma^2 + \beta\gamma)(\beta - \gamma) = (p^2 - 2q + q)\sqrt{p^2 - 4q}) = (p^2 - q)\sqrt{p^2 - 4q})$

Exercise LIII.

1. $x - 6\sqrt{x} + 9 = 25$; $\sqrt{x} - 3 = \pm 5$; $\sqrt{x} = 8$ or -2 $\therefore x = 64$ or 4

2. $\sqrt{x} - 4\sqrt[4]{x} + 4 = 1$; $\sqrt[4]{x} - 2 = \pm 1$; $\sqrt[4]{x} = 3$ or 1, $\therefore x = 81$ or 1

3. $x^4 - 14x^2 = -40$; $x^4 - 14x^2 + 49 = 9$; $x^2 - 7 = \pm 3$; $x^2 = 10$ or 4, $\therefore x = \pm 2$ or $\pm\sqrt{10}$

4. $x^3 + 14\sqrt{x^3} = 1107$; $x^3 + 14\sqrt{x^3} + 49 = 1156$; $x^{\frac{3}{2}} + 7 = \pm 34$; $x^{\frac{3}{2}} = 27$ or -41; $x^{\frac{1}{2}} = 3$ or $\sqrt[3]{-41}$; $\therefore x = 9$ or $\sqrt[3]{1681}$

5. $x - 2\sqrt{x+6} = 2$; $(x+6) - 2\sqrt{x+6} = 8$; $(x+6) - 2\sqrt{x+6} + 1 = 9$; $\sqrt{x+6} - 1 = \pm 3$; $\sqrt{x+6} = 4$ or -2; $x + 6 = 16$ or 4, $\therefore x = 10$ or -2

6. $x^4 - \frac{1}{2}x^2 = 248$; $x^4 - \frac{1}{2}x^2 + \frac{1}{16} = 248 + \frac{1}{16} = \frac{3969}{16}$; $x^2 - \frac{1}{4} = \pm\frac{63}{4}$; $x^2 = 16$ or $-\frac{62}{4}$, $\therefore x = \pm 4$ or $\pm\frac{1}{2}\sqrt{-62}$

7. $x^6 - 8x^3 = 513$; $x^6 - 8x^3 + 16 = 529$; $x^3 - 4 = \pm 23$; $x^3 = 27$ or -19, $\therefore x = 3$ or $\sqrt[3]{-19}$

8. $(x + 5) - \sqrt{x+5} = 6$; $(x + 5) - \sqrt{x+5} + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4}$; $\sqrt{x+5} - \frac{1}{2} = \pm\frac{5}{2}$; $\sqrt{x+5} = 3$ or -2; $x + 5 = 9$ or 4, $\therefore x = 4$ or -1

9. $\sqrt{x}(x^2+x-6)=0$, $\therefore \sqrt{x}=0$ and $\therefore x=0$

Also $x^2+x-6=0$, $\therefore x^2+x=6$; $x^2+x+\frac{1}{4}=\frac{25}{4}$; $x+\frac{1}{2}=\pm\frac{5}{2}$, $\therefore x=2$ or -3

10. Clearing of fractions $2x+3\sqrt{x}=16-x$; $3x-2\sqrt{x}=16$; $36x-24\sqrt{x}+4=192+4=196$; $6\sqrt{x}-2=\pm 14$; $6\sqrt{x}=-12$ or 16, $\therefore \sqrt{x}=-2$ or $\frac{8}{3}$, $\therefore x=4$ or $7\frac{1}{9}$

11. $\sqrt{x+21}+\sqrt[4]{x+21}=12$; $\sqrt{x+21}+\sqrt[4]{x+21}+\frac{1}{4}=\frac{49}{4}$; $\sqrt[4]{x+21}+\frac{1}{2}=\pm\frac{7}{2}$; $\sqrt[4]{x+21}=3$ or -4; $x+21=81$ or 256; $x=60$ or 235

12. $\sqrt{x}(x-2-\sqrt{x})=0$, $\therefore \sqrt{x}=0$, $\therefore x=0$. Also $x-2-\sqrt{x}=0$, $\therefore x-\sqrt{x}=2$; $x-\sqrt{x}+\frac{1}{4}=\frac{9}{4}$; $\sqrt{x}-\frac{1}{2}=\pm\frac{3}{2}$; $\sqrt{x}=2$ or -1, $\therefore x=4$ or 1

13. $\dfrac{x^5+x^4+2}{x^5-x^4}=\dfrac{x^3+x^2-2}{x^3-x^2}$. Then (by Art. 106 – VII)

$\dfrac{2x^5+2}{2x^4+2}=\dfrac{2x^3-2}{2x^2-2}$; $\dfrac{x^5+1}{x^4+1}=\dfrac{x^3-1}{x^2-1}$; $x^7-x^5+x^2-1=x^7-x^4+x^3-1$, or $x^2-x^5=x^3-x^4$; $x^5-x^4+x^3-x^2=0$; $x^4(x-1)+x^2(x-1)=0$; $(x-1)(x^4+x^2)=0$, $\therefore x-1=0$ or $x=1$. Also $x^4+x^2=0$, $\therefore x^4+x^2+\frac{1}{4}=\frac{1}{4}$; $x^2+\frac{1}{2}=\pm\frac{1}{2}$; $x^2=0$ or -1, $\therefore x=0$ or $\pm\sqrt{-1}$

14. $\dfrac{9(6-\sqrt{x})}{x+2\sqrt{x}}=\dfrac{7x^2-3x+4}{(6+\sqrt{x})(x+2\sqrt{x})}+\dfrac{23(x-2\sqrt{x})}{6+\sqrt{x}}$; multiplying by the denominator of the 2nd term we get

$9(36-x)=7x^2-3x+4+23(x^2-4x)$

or $324-9x=7x^2-3x+4+23x^2-92x$; $30x^2-86x=320$; or $x^2-\frac{43}{15}x=\frac{32}{3}$; $x^2-\frac{43}{15}x+(\frac{43}{30})^2=\frac{1849}{900}+\frac{9600}{900}=\frac{11449}{900}$; $x-\frac{43}{30}=\pm\frac{107}{30}$; $x=5$ or $-2\frac{2}{15}$

15. $x^3-3x^2+3x-9=0$; $x^2(x-3)+3(x-3)=0$; $(x-3)(x^2+3)=0$, $\therefore x-3=0$ or $x=3$; also $x^2+3=0$, $\therefore x^2=-3$; or $x=\pm\sqrt{-3}$

16. $(x-3)(x-4) = 2 - 2\sqrt{2}\sqrt{(x-1)(x-2)} + (x-1)(x-2)$; or $x^2 - 7x + 12 = 2 - 2\sqrt{2}\sqrt{x^2 - 3x + 2} + x^2 - 3x + 2$; or $-4x + 8 = -2\sqrt{2}\sqrt{x^2 - 3x + 2}$; $2x - 4 = \sqrt{2}\sqrt{x^2 - 3x + 2}$; $4x^2 - 16x + 16 = 2(x^2 - 3x + 2) = 2x^2 - 6x + 4$; or $2x^2 - 10x = -12$, $\therefore x^2 - 5x + \frac{25}{4} = \frac{25}{4} - \frac{24}{4} = \frac{1}{4}$. Hence $x - \frac{5}{2} = \pm\frac{1}{2}$, $\therefore x = 3$ or 2

17. $x^3 - 1 - 3x + 2 + 1 = 0$; or $x^3 - 1 - 3x + 3 = 0$; or $x^3 - 1 - 3(x-1) = 0$; or $(x-1)(x^2 + x + 1) - 3(x-1) = 0$; or $(x-1)(x^2 + x - 2) = 0$, $\therefore x - 1 = 0$ or $x = 1$. Also $x^2 + x - 2 = 0$, $\therefore x^2 + x = 2$; or $x^2 + x + \frac{1}{4} = \frac{9}{4}$; or $x + \frac{1}{2} = \pm\frac{3}{2}$, $\therefore x = 1$ or -2

18. Since $(\sqrt{x^2 + ax + b} + \sqrt{x^2 - ax + b})(\sqrt{x^2 + ax + b} - \sqrt{x^2 - ax + b}) = 2ax$, dividing these equals by the given equation we have $\sqrt{x^2 + ax + b} - \sqrt{x^2 - ax + b} = \dfrac{2ax}{c}$, and adding the given equation to this we get $2\sqrt{x^2 + ax + b} = \dfrac{2ax}{c} + c$; or, by squaring, $4x^2 + 4ax + 4b = \dfrac{4a^2x^2}{c^2} + 4ax + c^2$, $\therefore 4x^2 - \dfrac{4a^2x^2}{c^2} = c^2 - 4b$; or $x^2(4c^2 - 4a^2) = c^2(c^2 - 4b)$, $\therefore x = \pm\dfrac{c}{2}\sqrt{\dfrac{c^2 - 4b}{c^2 - a^2}}$

19. Reducing the terms of the first member to a common denominator and adding we get $\dfrac{x\sqrt{x} - x\sqrt{a-x} + x\sqrt{x} + x\sqrt{a-x}}{x - (a-x)} = \dfrac{b}{\sqrt{x}}$; or $\dfrac{2x\sqrt{x}}{2x - a} = \dfrac{b}{\sqrt{x}}$; or $2x^2 = 2bx - ab$, or $x^2 - bx = -\dfrac{ab}{2}$; $x^2 - bx + \dfrac{b^2}{4} = \dfrac{b^2}{4} - \dfrac{ab}{2} = \dfrac{b^2 - 2ab}{4}$; whence $x - \dfrac{b}{2} = \pm\frac{1}{2}\sqrt{b^2 - 2ab}$, or $x = \frac{1}{2}(b \pm \sqrt{b^2 - 2ab})$

20. Clearing of fractions we get $(\sqrt{x + 60} + \sqrt{x^2 + 9})^2 = 2\sqrt{x^3 + 60x^2 + 9x + 540} + 89$; that is $x + 60 + 2\sqrt{x^3 + 60x^2 + 9x + 540} + x^2 + 9 = 2\sqrt{x^3 + 60x^2 + 9x + 540} + 89$; or $x^2 + x = 20$; $x^2 + x + \frac{1}{4} = 20 + \frac{1}{4} = \frac{81}{4}$; $x + \frac{1}{2} = \pm\frac{9}{2}$ $\therefore x = 4$ or -5

21. $x^{12} - 1 = 0$; or $(x^6+1)(x^6-1) = 0$; or $(x^6+1)(x^3+1)(x^3-1) = 0$; or $(x^2+1)(x^4-x^2+1)(x+1)(x^2-x+1)(x-1)(x^2+x+1) = 0$ $\therefore$ we have, separately, $x^2+1=0$; or $x^2 = -1$, $\therefore x = \pm\sqrt{-1}$; Also $x^4 - x^2 + 1 = 0$; $x^4 - x^2 + \frac{1}{4} = -\frac{3}{4}$, $\therefore x^2 - \frac{1}{2} = \pm\frac{1}{2}\sqrt{-3}$, $\therefore x = \pm\sqrt{\frac{1}{2}(1\pm\sqrt{-3})}$; Also $x+1=0$, $\therefore x = -1$; Also $x^2 - x + 1 = 0$; or $x^2 - x = -1$, $\therefore x = \frac{1}{2}(1\pm\sqrt{-3})$; Also $x-1 = 0$, $\therefore x = 1$; Also $x^2+x+1=0$; or $x^2+x=-1$, $\therefore x = \frac{1}{2}(-1\pm\sqrt{-3})$

22. $x^3 - 6x^2 + 11x - 6 = 0$; multiplying by x we get $x^4 - 6x^3 + 11x^2 - 6x = 0$; or $x^4 - 6x^3 + 9x^2 + 2x^2 - 6x = 0$; or $(x^2-3x)^2 + 2(x^2-3x) = 0$, $\therefore (x^2-3x)^2 + 2(x^2-3x) + 1 = 1$; or $x^2 - 3x + 1 = \pm 1$, $\therefore x^2 - 3x = 0$ or -2; $x(x-3)^* = 0$, $\therefore x - 3 = 0$, or $x = 3$. Also $x^2 - 3x = -2$; whence $x = 2$ or 1

23. $x^3 - 4x^2 + x = 0$; or $x(x^2 - 4x + 1) = 0$, $\therefore x = 0$. Also $x^2 - 4x + 1 = 0$; whence $x = 2\pm\sqrt{3}$

24. $x^3 - 8x^2 + 11x + 20 = 0$; multiplying by x we get $x^4 - 8x^3 + 11x^2 + 20x = 0$; or $x^4 - 8x^3 + 16x^2 - 5x^2 + 20x = 0$; or $(x^2-4x)^2 - 5(x^2-4x) = 0$; $(x^2-4x)^2 - 5(x^2-4x) + \frac{25}{4} = \frac{25}{4}$, $\therefore x^2 - 4x - \frac{5}{2} = \pm\frac{5}{2}$; or $x^2 - 4x = 5$ or 0; $x(x-4)^* = 0$, $\therefore x = 4$. Also $x^2 - 4x = 5$, $\therefore x = 5$ or -1

25. $\dfrac{x+a-b+b}{x+b} = \left(\dfrac{2x+a+c+b-b}{2x+b+c}\right)^2$; or $\dfrac{x+b+a-b}{x+b} = \left(\dfrac{2x+b+c+a-b}{2x+b+c}\right)^2$; or $1 + \dfrac{a-b}{x+b} = \left(1 + \dfrac{a-b}{2x+b+c}\right)^2$;

or $1 + \dfrac{a-b}{x+b} = 1 + \dfrac{2(a-b)}{2x+b+c} + \dfrac{(a-b)^2}{(2x+b+c)^2}$; or $\dfrac{1}{x+b} - \dfrac{2}{2x+b+c} = \dfrac{a-b}{(2x+b+c)^2}$; or $\dfrac{2x+b+c-2x-2b}{(x+b)(2x+b+c)} = \dfrac{a-b}{(2x+b+c)^2}$;

or $\dfrac{c-b}{x+b} = \dfrac{a-b}{2x+b+c}$; or $2cx - 2bx - b^2 + c^2 = ax - bx + ab - b^2$;

or $(a+b-2c)x = c^2 - ab$; or $x = \dfrac{c^2-ab}{a+b-2c}$

26. $3x^3 - 14x^2 + 21x - 10 = 0$; multiplying by $3x$ we have $9x^4 - 42x^3 + 63x^2 - 30x = 0$; or $9x^4 - 42x^3 + 49x^2 + 14x^2 - 30x = 0$; or adding $x^2 - 5x$ to each side $(9x^4 - 42x^3 + 49x^2) + (15x^2 - 35x) = x^2 - 5x$; or $(3x^2 - 7x)^2 + 5(3x^2 - 7x) + \frac{25}{4} = x^2 - 5x + \frac{25}{4}$, $\therefore 3x^2 - 7x + \frac{5}{2} = \pm (x - \frac{5}{2})$

Then $3x^2 - 7x + \frac{5}{2} = x - \frac{5}{2}$	Or $3x^2 - 7x + \frac{5}{2} = \frac{5}{2} - x$
that is $3x^2 - 8x = -5$	that is $3x^2 - 6x = 0$
whence $x = 1\frac{2}{3}$ or 1	whence $x = 2$ or 0*

27. Assume $\sqrt[3]{x} + \sqrt[3]{a} = \sqrt[3]{n}$, then cubing each side we have $x + 3x^{\frac{2}{3}}a^{\frac{1}{3}} + 3x^{\frac{1}{3}}a^{\frac{2}{3}} + a = n$; or $x + a + 3\sqrt[3]{ax}(\sqrt[3]{x} + \sqrt[3]{a}) = n$; or $x + a + 3\sqrt[3]{anx} = n$ since $\sqrt[3]{x} + \sqrt[3]{a} = \sqrt[3]{n}$. But comparing this with the given equation we see that $n = b$, $\therefore \sqrt[3]{n} = \sqrt[3]{b}$, $\therefore \sqrt[3]{x} + \sqrt[3]{a} = \sqrt[3]{b}$; or $\sqrt[3]{x} = \sqrt[3]{b} - \sqrt[3]{a}$, $\therefore x = (\sqrt[3]{b} - \sqrt[3]{a})^3$

28. $(4x^2 - 9x) - (4x^2 - 9x + 11)^{\frac{1}{2}} = -5$, or adding 11 to each side we have $(4x^2 - 9x + 11) - (4x^2 - 9x + 11)^{\frac{1}{2}} = 6$; or completing the square $(4x^2 - 9x + 11) - (4x^2 - 9x + 11)^{\frac{1}{2}} + \frac{1}{4} = \frac{25}{4}$; $\therefore (4x^2 - 9x + 11)^{\frac{1}{2}} - \frac{1}{2} = \pm \frac{5}{2}$; or $(4x^2 - 9x + 11)^{\frac{1}{2}} = 3$ or -2, $\therefore 4x^2 - 9x + 11 = 9$ or 4, Then $4x^2 - 9x = -2$, whence $x = 2$ or $\frac{1}{4}$;

Also $4x^2 - 9x = -7$, whence $x = \dfrac{1}{8}(9 \pm \sqrt{-31})$

29. Completing the square we have $(x + 6)^2 + 2x^{\frac{1}{2}}(x + 6) + x = 138 + x + x^{\frac{1}{2}}$, and taking the square root, $x + 6 + \sqrt{x} = \pm\sqrt{(138 + x^{\frac{1}{2}} + x)}$; or $(x + \sqrt{x}) + 6 = \pm\sqrt{x + x^{\frac{1}{2}} + 138}$; squaring, we have $(x + \sqrt{x})^2 + 12(x + \sqrt{x}) + 36 = (x + x^{\frac{1}{2}}) + 138$; or $(x + \sqrt{x})^2 + 11(x + \sqrt{x}) = 102$; or $(x + \sqrt{x})^2 + 11(x + \sqrt{x}) + \frac{121}{4} = 102 + \frac{121}{4} = \frac{529}{4}$, $\therefore x + \sqrt{x} + \frac{11}{2} = \pm \frac{23}{2}$, $\therefore x + \sqrt{x} = 6$, whence $x = 4$ or 9, or $x + \sqrt{x} = -17$, whence $\sqrt{x} = \frac{1}{2}(-1 \pm \sqrt{-67})$, and $\therefore x = \frac{1}{2}(-33 \mp \sqrt{-67})$

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*We throw away the root $x = 0$ because it arises from the x by which we multiplied each side of the equation in the solution, and is consequently not a root of the given equation.

30. $x^4 - 4x^3 + 6x^2 - 4x + 1 = 6$, or extracting the square root $x^2 - 2x + 1 = \pm\sqrt{6}$; and again taking the sq. root $x - 1 = \pm\sqrt{\pm\sqrt{6}}$, whence $x = 1 \pm\sqrt{\pm\sqrt{6}}$

31. Squaring we have $4x^2 - 4x^6 = a^2 - 2a^2x^4 + a^2x^8$, and dividing by a^2x^4 we get, $\frac{4}{a^2x^2} - \frac{4x^2}{a^2} = \frac{1}{x^4} + 2 + x^4$; or $\frac{4}{a^2x^2} - \frac{4x^2}{a^2} - 4$ $= \frac{1}{x^4} - 2 + x^4$; or $\left(x^4 - 2 + \frac{1}{x^4}\right) + \left(\frac{4x^2}{a^2} - \frac{4}{a^2x^2}\right) = -4$;

or $\left(x^2 - \frac{1}{x^2}\right)^2 + \frac{4}{a^2}\left(x^2 - \frac{1}{x^2}\right) = -4$;

or $\left(x^2 - \frac{1}{x^2}\right)^2 + \frac{4}{a^2}\left(x^2 - \frac{1}{x^2}\right) + \frac{4}{a^4} = \frac{4}{a^4} - 4 = \frac{4 - 4a^4}{a^4} = \frac{4}{a^4}(1 - a^4)$

$\therefore x^2 - \frac{1}{x^2} + \frac{2}{a^2} = \pm\frac{2}{a^2}\sqrt{1 - a^4}$ $\therefore x^2 - \frac{1}{x^2} = -\frac{2}{a^2}(1 \mp \sqrt{1 - a^4})$

Let $-\frac{2}{a^2}(1 \mp \sqrt{1 - a^4})$ be represented by $2b^2$, then we have $x^2 - \frac{1}{x^2} = 2b^2$; or $x^4 - 2b^2x^2 = 1$; or $x^4 - 2b^2x^2 + b^4 = 1 + b^4$; or $x^2 - b^2 = \pm\sqrt{1 + b^4}$, $\therefore x^2 = b^2 \pm \sqrt{1 + b^4}$ (A)

But $2b^2 = -\frac{2}{a^2}(1 \mp \sqrt{1 - a^4})$, $\therefore b^2 = -\frac{1}{a^2}(1 \mp \sqrt{1 - a^4})$

$\therefore b^4 = \left\{-\frac{1}{a^2}(1 \mp \sqrt{1 - a^4})\right\}^2 = \frac{1}{a^4}(1 \mp 2\sqrt{1 - a^4} + 1 - a^4)$

$= \frac{1}{a^4}(2 \mp 2\sqrt{1 - a^4} - a^4)$ $\therefore b^4 + 1 = 1 + \frac{1}{a^4}(2 \mp 2\sqrt{1 - a^4} - a^4)$

$= 1 + \frac{1}{a^4}(2 \mp 2\sqrt{1 - a^4}) - \frac{a^4}{a^4} = \frac{1}{a^4}(2 \mp 2\sqrt{1 - a^4})$ $\therefore 1 + b^4$

$= \frac{2}{a^4}(1 \mp \sqrt{1 - a^4})$ Substituting these values for b^2 and $1 + b^4$ in equation A, and then extracting the square root we have

$x = \sqrt{-\frac{1}{a^2}(1 \mp \sqrt{1 - a^4}) \pm \sqrt{\frac{2}{a^4}(1 \mp \sqrt{1 - a^4})}}$ or using only the upper signs

$= \pm\frac{1}{a}\left\{\sqrt{-1 + \sqrt{1 - a^4} + \sqrt{2(1 \mp \sqrt{1 - a^4})}}\right\}$

or $x = \pm \dfrac{1}{a}\left\{ -1 + \sqrt{1 - a^4} + \sqrt{2 - 2\sqrt{1 - a^4}}\right\}^{\frac{1}{2}}$

$= \pm \dfrac{1}{a}\left\{ -1 + \sqrt{1 - a^4} + \sqrt{(1 + a^2) - 2\sqrt{1 - a^4} + (1 - a^2)}\right\}^{\frac{1}{2}}$

$\therefore\ x = \pm \dfrac{1}{a}\left\{ -1 + \sqrt{1 - a^4} + \sqrt{(\sqrt{1 + a^2} - \sqrt{1 - a^2})^2}\right\}^{\frac{1}{2}}$

$= \pm \dfrac{1}{a}\left\{ -1 + \sqrt{1 - a^4} + \sqrt{1 + a^2} - \sqrt{1 - a^2}\right\}^{\frac{1}{2}}$

$= \pm \dfrac{1}{a}\left\{\sqrt{1 - a^4} - \sqrt{1 - a^2} + \sqrt{1 + a^2} - 1\right\}^{\frac{1}{2}}$

$= \pm \dfrac{1}{a}\left\{(\sqrt{1 + a^2} - 1)(\sqrt{1 - a^2} + 1)\right\}^{\frac{1}{2}}$

32. $\{(x - 2)^2 - x\}^2 - \{(x - 2)^2 - x\} = 90$

$\therefore\ \{(x - 2)^2 - x\}^2 - \{(x - 2)^2 - x\} + \frac{1}{4} = \frac{361}{4}$;

or $\{(x - 2)^2 - x\} - \frac{1}{2} = \pm \frac{19}{2}$, $\therefore\ (x - 2)^2 - x = 10$ or -9

that is $x^2 - 4x + 4 - x = 10$, whence $x = 6$ or -1;

or $x^2 - 4x + 4 - x = -9$, whence $x = \frac{1}{2}(5 \pm 3\sqrt{-3})$

33. Dividing through by x^2 we have $ax^2 + bx + c + \dfrac{b}{x} + \dfrac{a}{x^2} = 0$;

or $\left(ax^2 + \dfrac{a}{x^2}\right) + \left(bx + \dfrac{b}{x}\right) + c = 0$; or $a\left(x^2 + \dfrac{1}{x^2}\right) + b\left(x + \dfrac{1}{x}\right) + c = 0$.

Let $x + \dfrac{1}{x} = y$; then $x^2 + \dfrac{1}{x^2} = y^2 - 2$, and substituting these values for x we have $a(y^2 - 2) + by = -c$; or $ay^2 + by = 2a - c$,

whence $y = \dfrac{-b \pm \sqrt{8a^2 + b^2 - 4ac}}{2a}$ $\therefore\ x + \dfrac{1}{x} = \dfrac{-b \pm \sqrt{8a^2 + b^2 - 4ac}}{2a}$

that is $2ax^2 + (b \mp \sqrt{8a^2 + b^2 - 4ac})x = -2a$, whence

$$x = \frac{\pm \sqrt{8a^2 + b^2 - 4ac} - b \pm \sqrt{-8a^2 + 2b^2 - 4ac \mp 2b\sqrt{8a^2 + b^2 - 4ac}}}{4a},$$

NOTE.—An equation such as the above, in which the coefficients following the middle term are the same as those preceding it but reversed in order, is called a *recurring equation*. The above solution affords a general method for solving such *recurring biquadratic equations*.

34. $\sqrt{\left(x^2-\frac{a^4}{x^2}\right)}-\frac{x^2}{a}=-\sqrt{\left(a^2-\frac{a^4}{x^2}\right)}$, squaring both sides we have $x^2-\frac{a^4}{x^2}+\frac{x^4}{a^2}-\frac{2x^2}{a}\left(x^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}=a^2-\frac{a^4}{x^2}$;

or $x^2-\frac{2x^2}{a}\left(x^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}+\left(\frac{x^4}{a^2}-a^2\right)=0$;

or $x^2-\frac{2x}{a}(x^4-a^4)^{\frac{1}{2}}+\frac{x^4-a^4}{a^2}=0$; or taking the square root we get, $x-\frac{\sqrt{(x^4-a^4)}}{a}=0$; or transposing and squaring, $x^2=\frac{x^4-a^4}{a^2}$;

or $a^2x^2=x^4-a^4$; or $x^4-a^2x^2=a^4$; or $x^4-a^2x^2+\frac{a^4}{4}=\frac{5a^4}{4}$

$\therefore x^2=\frac{a^2}{2}(1\pm\sqrt{5})$, whence $x=\pm a\sqrt{\tfrac{1}{2}(1\pm\sqrt{5})}$

35. $\sqrt{2x+4}-2\sqrt{2-x}=\frac{2\{(2x+4)-4(2-x)\}}{\sqrt{9x^2+16}}$; or factoring the second member, we have $\sqrt{2x+4}-2\sqrt{2-x}$ $=\frac{2\{(\sqrt{2x+4}-2\sqrt{2-x})(\sqrt{2x+4}+2\sqrt{2-x})\}}{\sqrt{9x^2+16}}$

Then dividing each side by $\sqrt{2x+4}-2\sqrt{2-x}$ we have $1=\frac{2(\sqrt{2x+4}+2\sqrt{2-x})}{\sqrt{9x^2+16}}$; or $\sqrt{9x^2+16}=2\{\sqrt{2x+4}+2\sqrt{2-x}\}$

Now squaring each side, we get $9x^2+16=48-8x+16\sqrt{8-2x^2}$

$\therefore x^2+8x=32-8x^2+16\sqrt{8-2x^2}$; or $x^2+8x=4(8-2x^2)+16\sqrt{8-2x^2}$

or $x^2+8x+16=4(8-2x^2)+16\sqrt{8-2x^2}+16$ $\therefore x+4=\pm(2\sqrt{8-2x^2}+4)$

$\therefore x=2\sqrt{8-2x^2}$, or $x^2=4(8-2x^2)$; whence $x=\pm\frac{4}{3}\sqrt{2}$

Also $x=-2\sqrt{8-2x^2}-8$, or $x^2+16x+64=4(8-2x^2)$; or $9x^2+16x=-32$, whence $x=-\frac{8}{9}\pm\frac{4}{9}\sqrt{-14}=-\frac{4}{9}(2\mp\sqrt{-14})$

And by equating the rejected factor $\sqrt{2x+4}-2\sqrt{2-x}=0$ we obtain the remaining root $x=\frac{2}{3}$

36. $\dfrac{2x^2+1+x\sqrt{4x^2+3}}{2x^2+3+x\sqrt{4x^2+3}} = \dfrac{a}{1}$, whence Algebra, Article 106

$\dfrac{2}{2x^2+3+x\sqrt{4x^2+3}} = \dfrac{1-a}{1} = 1-a$, $\therefore 2x^2+3+x\sqrt{4x^2+3} = \dfrac{2}{1-a}$;

or $x\sqrt{4x^2+3} = \dfrac{2}{1-a} - 3 - 2x^2 = \dfrac{3a-1}{1-a} - 2x^2$; squaring

each side we have $4x^4+3x^2 = \left(\dfrac{3a-1}{1-a}\right)^2 - 4x^2\left(\dfrac{3a-1}{1-a}\right) + 4x^4$

$\therefore 3x^2 + 4x^2\left(\dfrac{3a-1}{1-a}\right) = \left(\dfrac{3a-1}{1-a}\right)^2$; or $x^2\left(3+\dfrac{12a-4}{1-a}\right)$

$= x^2\left(\dfrac{9a-1}{1-a}\right) = \dfrac{(3a-1)^2}{(1-a)^2}$, $\therefore x^2(9a-1) = \dfrac{(3a-1)^2}{1-a}$

$\therefore x = \dfrac{3a-1}{\sqrt{(1-a)(9a-1)}}$

37. $\{(x-1)(x-4)\}\{(x-2)(x-3)\} = 8$; $\{x^2-5x+4\}\{x^2-5x+6\} = 8$ $\therefore \{(x^2-5x)+4\}\{(x^2-5x)+6\}$, that is $(x^2-5x)^2+10(x^2-5x)+24=8$ or $(x^2-5x)^2+10(x^2-5x)+25=9$, $\therefore x^2-5x+5=\pm 3$ $\therefore x^2-5x=-2$, whence $x=\frac{1}{2}(5\pm\sqrt{17})$. Also $x^2-5x=-8$, whence $x=\frac{1}{2}(5\pm\sqrt{-7})$

38. $\{(x-1)(x-8)\}\{(x-2)(x-7)\}\{(x-3)(x-6)\}\{(x-4)(x-5)\}$ $=\{(x^2-9x)+8\}\{(x^2-9x)+14\}\{(x^2-9x)+18\}\{(x^2-9x)+20\}$ $=(x^2-9x)(17x^2-153x+230)+401$. For x^2-9x write y, then we have $(y+8)(y+14)(y+18)(y+20)=17y^2+230y+401$; that is $y^4+60y^3+1308y^2+12176y+40320=17y^2+230y+401$, subtracting from each side $8y^2+176y+320$ we have $y^4+60y^3+1300y^2+12000y+40000=9y^2+54y+81$, or taking the square root of each side $y^2+30y+200=\pm(3y+9)$ $\therefore y^2+27y=-191$, whence $y=\frac{1}{2}(-27\pm\sqrt{-35})$
Also $y^2+33y=-209$, whence $y=\frac{1}{2}(-33\pm\sqrt{253})$
But $y=x^2-9x$, $\therefore x^2-9x=\frac{1}{2}(-27\pm\sqrt{-35})$, whence $x=\frac{1}{2}(9\pm\sqrt{27\pm\sqrt{-33}}$
Also $x^2-9x=\frac{1}{2}(-33\pm\sqrt{253})$, whence $x=\frac{1}{2}(9\pm\sqrt{15\pm 2\sqrt{253}})$

39. Multiplying as indicated we have $x^3 - 6x^2 + 11x - 6 = x^3 + 6x^2 + 11x + 6$, whence $12x^2 + 12 = 0$, $\therefore x = \pm\sqrt{-1}$

40. Reducing as indicated by the question we have
$x + 1 - 5\sqrt{x+1} + 6 + 5\sqrt{1 + x - 6\sqrt{x+1} + \sqrt{x+1} - 1} = 0$;
or $(x - 5\sqrt{x+1}) + 5\sqrt{x - 5\sqrt{x+1}} = -7$; or completing the square $(x - 5\sqrt{x+1}) + 5(x - 5\sqrt{x+1})^{\frac{1}{2}} + \frac{25}{4} = -\frac{3}{4}$, whence $(x - 5\sqrt{x+1})^{\frac{1}{2}} = \dfrac{-5 \pm \sqrt{-3}}{2}$, $\therefore x - 5\sqrt{x+1} = \frac{1}{2}(11 \pm 5\sqrt{-3}) = a$, suppose:
Then $x - 5\sqrt{x+1} = a$; or $x - a = 5\sqrt{x+1}$; or $x^2 - 2ax + a^2 = 25x + 25$; or $x^2 - (2a + 25)x = 25 - a^2$, $\therefore x = \frac{1}{2}(2a + 25 \pm 5\sqrt{4a + 29})$
But $a = \frac{1}{2}(11 \pm 5\sqrt{-3})$ by supposition
$\therefore x = \frac{1}{2}\{11 \pm 5\sqrt{-3} + 25 \pm 5\sqrt{22 \pm 10\sqrt{-3} + 29}\}$
$= 18 \pm \frac{5}{2}(\sqrt{-3} + \sqrt{51 \pm 10\sqrt{-3}})$

41. Arranging the given quantities, we have
$(4x^4 - 8x^3 - 4x^2 + 3x - 1) - 2(2x^2 - 2x + 1)\sqrt{4x^4 - 8x^3 - 4x^2 + 3x - 1} + (4x^4 - 8x^3 + 8x^2 - 4x + 1) = 0$, and taking the square root $\sqrt{4x^4 - 8x^3 - 4x^2 + 3x - 1} - (2x^2 - 2x + 1) = 0$; or transposing and squaring $4x^4 - 8x^3 - 4x^2 + 3x - 1 = 4x^4 - 8x^3 + 8x^2 - 4x + 1$
$\therefore 12x^2 - 7x = -2$, whence $x = \frac{1}{24}(7 \pm \sqrt{-47})$

42. Multiplying through by ax to clear of fractions
$a^2bx^{-1} + 2a^3c^{-1}x^2 - 2ab + 2a^2c^{-1}x^3 - 2bx = ac^{-1}x(x^3 - a^{-2}bcx + a^3)$
multiplying now by c we have
$a^2bcx^{-1} + 2a^3x^2 - 2abc + 2a^2x^3 - 2bcx = ax^4 - a^{-1}bcx^2 + a^4x$;
or transposing and changing signs
$ax^4 - 2a^2x^3 - 2a^3x^2 + a^4x - a^{-1}bcx^2 + 2bcx + 2abc - a^2bcx^{-1} = 0$
dividing through by ax we now have
$x^3 - 2ax^2 - 2a^2x + a^3 - bc(a^{-2}x - 2a^{-1} - 2x^{-1} + ax^{-2}) = 0$;
or $a^2x^2(a^{-2}x - 2a^{-1} - 2x^{-1} + ax^{-2}) - bc(a^{-2}x - 2a^{-1} - 2x^{-1} + ax^{-2}) = 0$
$\therefore (a^2x^2 - bc)(a^{-2}x - 2a^{-1} - 2x^{-1} + ax^{-2}) = 0$, or factoring the first member $(a^2x^2 - bc)(a^{-2}x^{-1} + a^{-1}x^{-2})(x^2 - 3ax + a^2) = 0$
$\therefore a^2x^2 - bc = 0$, whence $x = \pm a^{-1}\sqrt{bc}$

Also $a^{-2}x^{-1} + a^{-1}x^{-2} = 0$; or $\frac{1}{a^2x} = -\frac{1}{ax^2}$; or $ax^2 = -a^2x$, whence $x = -a$

Also $x^2 - 3ax + a^2 = 0$; or $x^2 - 3ax = -a^2$, whence $x = \frac{a}{2}(3 \pm \sqrt{5})$

43. Add x^4 to each side, then

$x^4 + 8x^3 + 22x^2 + 24x + 9 = x^4$, and taking the square root $x^2 + 4x + 3 = \pm x^2$, $\therefore 4x = -3$, whence $x = -\frac{3}{4}$; also $2x^2 + 4x = -3$, whence $x = -\frac{1}{2}(2 \mp \sqrt{-2})$

44. Changing signs, adding $(4 - 2x^4)$ to each side, and arranging we have $x^4 - 4x^2 + 4 = 4x^4 - 4x^3 + 13x^2 - 6x + 9$, and now extracting the square root $\pm(x^2 - 2) = 2x^2 - x + 3$, $\therefore 2x^2 - x + 3 = x^2 - 2$; whence $x^2 - x = -5$, and $\therefore x = \frac{1}{2}(1 \pm \sqrt{-19})$

Also $3x^2 - x = -1$, whence $x = \frac{1}{2}(1 \pm \sqrt{-11})$

45. $\frac{x^2 + 2x(\sqrt{3} - \sqrt{5}) + (8 - 2\sqrt{15})}{x - \sqrt{3} + \sqrt{5}} - \frac{x^2 - 2x(\sqrt{3} - \sqrt{5}) + (8 - 2\sqrt{15})}{x + \sqrt{3} - \sqrt{5}}$
$= x^2 - 8 - \sqrt{15}$

That is $\frac{(x + \sqrt{3} - \sqrt{5})^2}{x - \sqrt{3} + \sqrt{5}} - \frac{(x - \sqrt{3} + \sqrt{5})^2}{x + \sqrt{3} - \sqrt{5}} = x^2 + 8 - 2\sqrt{15}$; or clearing of fractions

$(x + \sqrt{3} - \sqrt{5})^3 - (x - \sqrt{3} + \sqrt{5})^3 = (x^2 + 8 - 2\sqrt{15})(x^2 - 8 + 2\sqrt{15})$;
or $6x^2(\sqrt{3} - \sqrt{5}) + 2(\sqrt{3} - \sqrt{5})^3 = \{x^2 + (\sqrt{3} - \sqrt{5})^2\}\{x^2 - (\sqrt{3} - \sqrt{5})^2\}$;
Let $\sqrt{3} - \sqrt{5} = a$, then $x^4 - 6ax^2 = a^4 - 2a^3$, whence $x = \pm\sqrt{3a \pm a\sqrt{a^2 + 2a + 9}}$ where $a = \sqrt{3} - \sqrt{5}$

Exercise LIV.

1. $(x + y)(x - y) = 45$, but $x - y = 5$ $\therefore 5(x + y) = 45$, or $x + y = 9$ and $x - y = 5$ $\therefore 2x = 14$, &c.

2. $(x + y)(x - y) = 105$, but $x + y = 21$ $\therefore 21(x - y) = 105$, or $x - y = 5$ $\therefore 2x = 26$, &c.

3. $x^2 + 2xy + y^2 = 81$, but $x^2 + y^2 = 41$ $\therefore$ $2xy = 40$, and $\therefore$ $x^2 - 2xy + y^2 = 1$, whence $x - y = \pm 1$, $\therefore$ $2x = 10$ or 8, &c.

4. $x^2 - 2xy + y^2 = 225$, but $x^2 + y^2 = 113$ $\therefore$ $2xy = -112$ $\therefore$ $x^2 + 2xy + y^2 = 113 - 112 = 1$, whence $x + y = \pm 1$ and $x - y = 15$ $\therefore$ $2x = 16$ or 14, &c.

5. $x^2 + y^2 = 89$ and $x = \dfrac{40}{y}$ $\therefore$ $\dfrac{40^2}{y^2} + y^2 = 89$; or $y^4 - 89y^2 = -1600$;

$y^4 - 89y^2 + (\frac{89}{2})^2 = -1600 + \frac{7921}{4} = \frac{1521}{4}$ $\therefore$ $y^2 - \frac{89}{2} = \pm\frac{39}{2}$, whence $y = \pm 8$ or ± 5. And $x = \dfrac{40}{y} = \dfrac{40}{\pm 8}$; or $\dfrac{40}{\pm 5} = \pm 5$ or ± 8

6. $x^2 - y^2 = 55$, and $x = \dfrac{72}{3y} = \dfrac{24}{y}$ $\therefore$ $\dfrac{24^2}{y^2} - y^2 = 55$; or $y^4 + 55y^2 = 576$ whence $y^2 = 9$ or -64, and $\therefore$ $y = \pm 3$ or $\pm 8\sqrt{-1}$

And $x = \dfrac{24}{y} = \dfrac{24}{\pm 3}$; or $\dfrac{24}{\pm 8\sqrt{-1}} = \pm 8$ or $\mp 3\sqrt{-1}$

7. $x^2 + 3y^2 = 148$, and $y = 24 - 2x$ $\therefore$ $x^2 + 3(24 - 2x)^2 = 148$; or $x^2 - 1728 - 288x + 12x^2 = 148$; or $13x^2 - 288x = -1580$, whence $x = 12\frac{2}{13}$ or 10

And $y = 24 - 2x = (24 - 24\frac{4}{13})$ or $(24 - 20) = -\frac{4}{13}$ or 4

8. $3x^2 - 2y^2 = 115$, and $x = \dfrac{2 + 3y}{2}$ $\therefore$ $3\left(\dfrac{2 + 3y}{2}\right)^2 - 2y^2 = 115$; or $19y^2 + 36y = 448$, whence $y = 4$ or $-5\frac{17}{19}$

And $x = \dfrac{2 + 3y}{2} = \dfrac{2 + 12}{2}$; or $\dfrac{2 - 17\frac{13}{19}}{2} = 7$ or $-7\frac{16}{19}$

9. $4x^2 + 3y^2 = 511$, and $x = 9 - \dfrac{2y}{3}$ $\therefore$ $4\left(9 - \dfrac{2y}{3}\right)^2 + 3y^2 = 511$, or $43y^2 - 432y = 1683$; whence $y = 13\frac{2}{43}$ or -3

And $x = 9 - \dfrac{2y}{3} = \left(9 - \dfrac{26\frac{4}{43}}{3}\right)$ or $(9 + \frac{6}{3}) = \frac{13}{43}$ or 11

10. $x^3 - y^3 = 26$; also from 2nd equat. $x^3 - 3x^2y + 3xy^2 - y^3 = 8$ $\therefore$ by subtraction $3x^2y - 3xy^2 = 18$; or $xy(x - y) = 6$, but $x - y = 2$ $\therefore$ $2xy = 6$ or $xy = 3$. Then $xy = 3$ and $x = 2 + y$ $\therefore$ $y(2 + y) = 3$ or $y^2 + 2y = 3$, whence $y = 1$ or -3. And $x = 2 + y = 3$ or -1

11. $x + y = 4 \therefore (x+y)^2 = 16 \therefore x^3 + y^3 = 16$, and from 1st equat. $x^3 + 3x^2y + 3xy^2 + y^3 = 64$, $\therefore$ by subtraction $3x^2y + 3xy^2 = 48$; or $xy(x+y) = 16$, but $x + y = 4 \therefore xy = 4 \therefore y(4-y) = 4$ or $y^2 - 4y = -4$, whence $y = 2$, and $x = 4 - y = 4 - 2 = 2$

12. Squaring tbe 1st equat. $\sqrt[4]{x} + 2\sqrt[8]{xy} + \sqrt[4]{y} = 9$, but $4\sqrt[8]{xy} = 8$ $\therefore$ subtracting we have $\sqrt[4]{x} - 2\sqrt[8]{xy} + \sqrt[4]{y} = 1$; whence $\sqrt[8]{x} - \sqrt[8]{y} = \pm 1$ and $\sqrt[8]{x} + \sqrt[8]{y} = 3$, $\therefore$ by addition $2\sqrt[8]{x} = 4$ or 2, $\therefore \sqrt[8]{x} = 2$ or 1, whence $x = 256$ or 1, &c.

13. $y^2 + 4x - 2y = 11$, and $x = 14 - 4y \therefore y^2 + 4(14 - 4y) - 2y = 11$, or $y^2 - 18y = -45$; whence $y = 15$ or 3, and $x = 14 - 4y = -46$ or 2

14. $2x^2 + xy - 5y^2 = 20$, and $x = \dfrac{3y+1}{2}$

$\therefore 2\left(\dfrac{3y+1}{2}\right)^2 + y\left(\dfrac{3y+1}{2}\right) - 5y^2 = 20$; or $2y^2 + 7y = 39$, whence $y = 3$ or $-6\frac{1}{2}$ and $x = \dfrac{3y+1}{2} = 5$ or $-9\frac{1}{4}$

15. $9x + 5y - 4xy = 0$, and $x = 2 + y \therefore 9(2+y) + 5y - 4y(2+y) = 0$, or $2y^2 - 3y = 9$; whence $y = 3$ or $-\frac{3}{2}$, and $x = 2 + y = 5$ or $\frac{1}{2}$

16. $x^2y^2 + 4xy + 4 = 100 \therefore xy + 2 = \pm 10$; whence $xy = 8$ or -12 From second equation $x = 6 - y \therefore y(6-y) = 8$ or -12 That is $y^2 - 6y = -8$, whence $y = 4$ or 2; and $y^2 - 6y = 12$, whence $y = 3 \pm\sqrt{21} \therefore x = 6 - y = 2$ or 4, or $3 \mp\sqrt{21}$

17. $9x^2 + 36xy - 85y^2 = 0$, and $x = 2 + y$ $\therefore 9(2+y)^2 + 36y(2+y) - 85y^2 = 0$. That is $10y^2 - 27y = 9$; whence $y = 3$ or $-\frac{3}{10}$, and $x = 2 + y = 5$ or $1\frac{7}{10}$

18. From second equation $x = \dfrac{12 + y^2}{y}$ and substituting this for x in the 1st equat. we get $\left(\dfrac{12+y^2}{y}\right)^2 + \left(\dfrac{12+y^2}{y}\right)y = 77$; or $\dfrac{144 + 24y^2 + y^4}{y^2} + 12 + y^2 = 77$; or $2y^4 - 41y^2 = -144$ $\therefore y^2 = 16$ or $\frac{18}{4}$, whence $y = \pm 4$ or $\pm\frac{3}{2}\sqrt{2}$

And $x = \dfrac{12 + y^2}{y} = \dfrac{28}{\pm 4}$ or $\dfrac{\frac{1}{4}\text{ of } 66}{\pm\frac{3}{2}\sqrt{2}}$; $= \pm 7$ or $\dfrac{33}{\pm 3\sqrt{2}}$; $= \pm 7$ or $\pm\dfrac{11\sqrt{2}}{2}$

19. Let $x = v + z$ and $y = v - z$
Then $x^2 + xy = (v + z)^2 + (v + z)(v - z) = 2v^2 + 2vz = 66$ (I)
Also $x^2 - y^2 = (v + z)^2 - (v - z)^2 = 4vz = 11 \therefore 2vz = \frac{11}{2}$ (II)
From (I) we get $2v^2 = 66 - \frac{11}{2} = \frac{121}{2} \therefore v^2 = \frac{121}{4}$ or $v = \pm \frac{11}{2}$
From (II) we get by substitution $z = \frac{11}{2} \div \pm 11 = \pm \frac{1}{2}$. Then $x = v + z = \pm \frac{11}{2} \pm \frac{1}{2} = \pm 6$. And $y = v - z = \pm 11 \mp \frac{1}{2} = \pm 5$

20. From 1st equat. by clearing of fractions $x^3 + y^3 = 18xy$ (I) and cubing the 2nd equat. we get $x^3 + 3x^2y + 3xy^2 + y^3 = 1728$ (II) and taking (I) from (II) we have $3x^2y + 3xy^2 = 1728 - 18xy$; or $xy(x + y) = 576 - 6xy$; or since $x + y = 12$, we have $12xy = 576 - 6xy \therefore 18xy = 576$, and hence $xy = 32$. Then $x = 12 - y$ $\therefore y(12 - y) = 32$, or $y^2 - 12y = -32$; whence $y = 8$ or 4. And $x = 12 - y = 4$ or 8

21. Let $x = v + z$ and $y = v - z$
Then $x^5 + y^5 = (v + z)^5 + (v - z)^5 = 2v^5 + 20v^3z^2 + 10vz^4 = 3368$; or $v^5 + 10v^3z^2 + 5vz^4 = 1684$. But $x + y = v + z + v - z = 2v = 8$ $\therefore v = 4$, and substituting this for v, $1024 + 640z^2 + 20z^4 = 1684$; whence $z^4 + 32z^2 = 33$, $\therefore z^2 = 1$ or -33 and $z = \pm 1$ or $\pm\sqrt{-33}$
Then $x = v + z = 4 \pm 1$; or $4 \pm \sqrt{-33} = 5$ or 3 or $4 \pm \sqrt{-33}$
$y = v - z = 4 \mp 1$; or $4 \mp \sqrt{-33} = 3$ or 5 or $4 \mp \sqrt{-33}$

22. From 1st equat. $x^3 + 3x^2y + 3xy^2 + y^3 = 343$, and $x^3 + y^3 = 133$ $\therefore 3x^2y + 3xy^2 = 210$; or $xy(x + y) = 70$, but $x + y = 7 \therefore xy = 10$ And $x = 7 - y \therefore y(7 - y) = 10$; whence $y = 5$ or 2, and $x = 2$ or 5

23. Let $x = v + z$ and $y = v - z$
Then $x^4 + y^4 = (v + z)^4 + (v - z)^4 = 2v^4 + 12v^2z^2 + 2z^4 = 97$
But $x - y = v + z - v + z = 2z = 1 \therefore z = \frac{1}{2}$
Hence $2v^4 + 3v^2 = 97 - \frac{1}{8} = 96\frac{7}{8}$; whence $v^2 = \frac{25}{4}$ or $-\frac{31}{4}$
$\therefore v = \pm \frac{5}{2}$ or $\pm \frac{1}{2}\sqrt{-31}$
Then $x = v + z = \pm\frac{5}{2} + \frac{1}{2}$; or $\pm \frac{1}{2}\sqrt{-31} + \frac{1}{2} = 3$ or -2 or $\frac{1}{2}(1 \pm \sqrt{-31})$
And $y = v - z = \pm\frac{5}{2} - \frac{1}{2}$; or $\pm \frac{1}{2}\sqrt{-31} - \frac{1}{2} = 2$ or -3 or $\frac{1}{2}(-1 \pm \sqrt{-31})$

24. Multiplying the 2nd equat. by 3 and adding it to the 1st equation we have $x^3 + 3x^2y + 3xy^2 + y^3 = 343$; whence $x + y = 7$ $\therefore x = 7 - y$. Also $x^2y + xy^2 = xy(x + y) = 7xy = 84$ $\therefore xy = 12$ $\therefore y(7 - y) = 12$, or $y^2 - 7y = -12$; whence $y = 4$ or 3, and $x = 7 - y = 3$ or 4

25. $x^2 + y^2 + x + y = 26$, adding $2xy$ to each side of the equat. we have $(x^2 + 2xy + y^2) + (x + y) = 26 + 2xy$, or completing the square $(x + y)^2 + (x + y) + \frac{1}{4} = 26\frac{1}{4} + 2xy$ $\therefore x + y + \frac{1}{2} = \pm\sqrt{26\frac{1}{4} + 2xy}$ or $x + y = \pm\sqrt{26\frac{1}{4} + 2xy} - \frac{1}{2}$, but $4(x + y) = 3xy$ $\therefore 3xy = \pm 4\sqrt{26\frac{1}{4} + 2xy} - 2$; transposing and squaring these we have $9x^2y^2 + 12xy + 4 = 420 + 32xy$ $\therefore 9x^2y^2 - 20xy = 416$; whence $xy = 8$ or $-\frac{52}{9}$ Then $4(x + y) = 3xy = 24$ $\therefore x + y = 6$, and $x = 6 - y$ $\therefore y(6 - y) = 8$, or $y^2 - 6y = -8$; whence $y = 4$ or 2, and $\therefore x = 2$ or 4. Also $y(6 - y) = -\frac{52}{9}$; whence $y = \frac{1}{6}(-13 \pm \sqrt{377})$ and $\therefore x = \frac{1}{6}(-13 \mp \sqrt{377})$

26. Clearing the first equation of fractions we have $5\{(x + y)^2 + (x - y)^2\} = 26(x^2 - y^2)$: *i. e.* $10x^2 + 10y^2 = 26x^2 - 26y^2$ Hence $36y^2 = 16x^2$, or $6y = \pm 4x$, or $y = \pm\dfrac{2x}{3}$; substituting this in the 2nd equation, we have $x^2 + \left(\pm\dfrac{2x}{3}\right)^2 = 52$; or $x^2 + \dfrac{4x^2}{9} = 52$ or $\dfrac{13x^2}{9} = 52$ $\therefore \dfrac{x^2}{9} = 4$; or $\dfrac{x}{3} = \pm 2$; or $x = \pm 6$. And $y = \pm\dfrac{2x}{3} = \pm 4$

27. $7y = 2x + 36$ $\therefore y = \dfrac{2x + 36}{7}$; substituting this in the first equation, we have $x + \dfrac{2x + 36}{7} = x^2$, or $7x^2 - 9x = 36$; whence $x = 3$ or $-1\frac{5}{7}$. And $y = \dfrac{2x + 36}{7} = 6$ or $4\frac{32}{49}$

28. Let $x = v + z$, and $y = v - z$; then we have from the first equation $2v^4 + 12v^2z^2 + 2z^4 = 14v^4 - 28v^2z^2 + 14z^4$, and this by transposition gives $40v^2z^2 = 12v^4 + 12z^4$; but $x + y = v + z + v - z$

$= 2v = m$, $\therefore v = \frac{1}{2}m$: substituting this for v, we have $10m^2z^2 = \frac{3}{4}m^4 + 12z^4$, or $6z^4 - 5m^2z^2 = -\frac{3m^4}{8}$; whence $z = \pm\frac{3m}{2\sqrt{3}}$ or $\pm\frac{m}{2\sqrt{3}}$

Then $x = v + z = \frac{1}{2}m \pm \frac{3m}{2\sqrt{3}}$; or $\frac{1}{2}m \pm \frac{m}{2\sqrt{3}} = \frac{m}{2}(1 \pm \sqrt{3})$; or $\frac{m}{2}(1 \pm \frac{1}{3}\sqrt{3})$

And $y = v - x = \frac{m}{2} \mp \frac{3m}{2\sqrt{3}}$; or $\frac{m}{2} \mp \frac{m}{2\sqrt{3}} = \frac{m}{2}(1 \mp \sqrt{3})$; or $\frac{m}{2}(1 \mp \frac{1}{3}\sqrt{3})$

29. Let $x = vy$; then $v^2y^2 + 2y^2 + vy^2 = 74$, or $y^2 = \frac{74}{v^2 + v + 2}$ and $2vy^2 + y^2 + 2v^2y^2 = 73$; or $y^2 = \frac{73}{2v^2 + 2v + 1}$ $\therefore \frac{74}{v^2 + v + 2} = \frac{73}{2v^2 + 2v + 1}$ or by reduction $v^2 + v = \frac{24}{25}$; whence $v = \frac{3}{5}$ or $-\frac{8}{5}$; $y^2 = \frac{74}{v^2 + v + 2}$ $= \frac{74}{\frac{9}{25} + \frac{3}{5} + 2} = 25$ $\therefore y = \pm 5$, also $y^2 = \frac{74}{\frac{64}{25} - \frac{8}{5} + 2} = 25$; whence as before $y = \pm 5$. Therefore $x = vy = \pm 5 \times \frac{3}{5}$; or $\pm 5 \times -\frac{8}{5} = \pm 3$ or ∓ 8

30. Adding the two equations together, we have $x^4 + 2x^2y^2 + y^4 = 169$, whence $x^2 + y^2 = \pm 13$; but $x^2 + 2x^2y^2 + y^2 = 85$ $\therefore$ by subtraction $2x^2y^2 = 72$ or 98 $\therefore x^2y^2 = 36$ or 49, and $xy = \pm 6$ or ± 7

Then $x = \frac{\pm 6}{y}$ or $\frac{\pm 7}{y}$, and substituting this for x, we have $\left(\pm\frac{6}{y}\right)^2 + y^2 = \pm 13$, that is $\frac{36}{y^2} + y^2 = \pm 13$; or $y^4 \mp 13y^7 = -36$, whence $y^2 = 9$ or 4, and $\therefore y = \pm 3$ or ± 2. (Impossible values being rejected.)

31. Let $x = vy$; then $3v^2y^2 + 2vy^2 - 4y^2 = 108$; or $y^2 = \frac{108}{3v^2 + 2v - 4}$

Also $v^2y^2 - 3vy^2 - 7y^2 = -81$; whence $y^2 = \frac{81}{3v - v^2 + 7}$

$\therefore \frac{108}{3v^2 + 2v - 4} = \frac{81}{3v - v^3 + 7}$ or by reduction, $13v^2 - 6v = 40$; whence $v = 2$ or $-\frac{20}{13}$

Then $y^2 = \frac{81}{3v - v^2 + 7}$; $= \frac{81}{6 - 4 + 7}$ or $\frac{81}{-\frac{60}{13} - \frac{400}{169} + 7}$; $= 9$ or 169×27 $\therefore y = \pm 3$ or $= \pm 39\sqrt{3}$

And $x = vy = \pm 3 \times 2$; or $\pm 3 \times - \frac{20}{13} = \pm 6$ or $\mp 4\frac{8}{13}$

Also $x = vy = \pm 39\sqrt{3} \times 2$; or $\pm 39\sqrt{3} \times - \frac{20}{13} = \pm 78\sqrt{3}$ or $\mp 60\sqrt{3}$

32. Factoring the first equation, we have $(y^2 - x^2) - (y + x) = 12$ $\therefore (y+x)(y-x-1) = 12$; or $y + x = \dfrac{12}{y-x-1}$

but $y + x = \dfrac{48}{(y-x)^2}$ $\therefore \dfrac{12}{y-x-1} = \dfrac{48}{(y-x)^2}$; or $\dfrac{1}{y-x-1} = \dfrac{4}{(y-x)^2}$; or $(y-x)^2 = 4(y-x) - 4$ $\therefore (y-x)^2 - 4(y-x) + 4 = 0$; whence $y - x = 2$, and $y + x = \dfrac{48}{(y-x)^2} = \frac{48}{4} = 12$, $\therefore y = 14$ or $2y = 7$, and $x = 5$

33. Transposing the first equation, we have

$$\frac{x^2}{y^2} + \frac{2x}{\sqrt{y}} + y + \frac{x}{y} + \sqrt{y} = 20; \text{ that is } \left(\frac{x}{y} + \sqrt{y}\right)^2 + \left(\frac{x}{y} + \sqrt{y}\right) = 20$$

$\therefore \dfrac{x}{y} + \sqrt{y} = 4$ or -5; that is $x + y^{\frac{3}{2}} = 4y$ or $-5y$

Taking $x + y^{\frac{3}{2}} = 4y = x + 8$, (by 2nd given equation) we have $y^{\frac{3}{2}} = 8$ $\therefore y = 4$, and $x = 4y - 8 = 8$

Taking $x + y^{\frac{3}{2}} = -5y$, and subtracting this from the 2nd given equat., we have $8 - y^{\frac{3}{2}} = 9y$ $\therefore 8 - 8y = y + y^{\frac{3}{2}}$; or $8(1-y) = y(1 + y^{\frac{1}{2}})$ and dividing each side by $1 + y^{\frac{1}{2}}$, we have $8(1 - y^{\frac{1}{2}}) = y$; that is $y + 8y^{\frac{1}{2}} = 8$, whence $y^{\frac{1}{2}} = -4 \pm 2\sqrt{6}$, and $\therefore y = 40 \mp 16\sqrt{6}$, and $x = 4y - 8 = 152 \mp 64\sqrt{6}$; also $1 + y^{\frac{1}{2}} = 0$ $\therefore y = 1$

34. $x^3 + y^3 = 35$ (I), $x^2 + y^2 = 13$ (II).

From (I) $(x + y)(x^2 - xy + y^2) = 35$; but $x^2 + y^2 = 13$ $\therefore (x+y)(13 - xy) = 35$ $\therefore x + y = \dfrac{35}{13 - xy}$, squaring we have $x^2 + 2xy + y^2 = \dfrac{1225}{169 - 26xy + x^2y^2}$; substituting (II) in this, we have $2xy = \dfrac{1225}{169 - 26xy + x^2y^2} - 13 = \dfrac{1225 - 2197 + 338xy - 13x^2y^2}{169 - 26xy + x^2y^2}$

$\therefore 2xy = \dfrac{338xy - 13x^2y^2 - 972}{169 - 26xy + x^2y^2}$; clearing of fractions, we have $2x^3y^3 - 39x^2y^2 + 972 = 0$; and factoring $(xy - 6)(2x^2y^2 - 27xy - 162)$

$= 0 \therefore xy - 6 = 0$ (III); and also $2x^2y^2 - 27xy - 162 = 0$ (IV)
From (III) $xy = 6$, and from (IV) $xy = 18$ or $4\frac{1}{2} \therefore 2xy = 12$, or 36, or 9
Hence adding these to (II) and extracting the square root, we have $x + y = \pm 5$, or ± 7, or $\pm\sqrt{22}$; similarly subtracting these from (II), and then extracting the square root $x - y = \pm 1$, or $\pm\sqrt{-23}$, or ± 2. Hence by addition and subtraction we have
$x = \pm 3$; or $\pm\frac{1}{2}(7 + \sqrt{-23})$; or $\pm\frac{1}{2}(2 + \sqrt{22})$
$y = \pm 2$; or $\pm\frac{1}{2}(7 - \sqrt{-23})$; or $\pm\frac{1}{2}(\sqrt{22} - 2)$

Otherwise, thus:
Let $x = v + z$, and $y = v - z$
Then from (I) $2v^3 + 6vz^2 = 35$ (III), and from (II) $2v^2 + 2z^2 = 13$ (IV)
Multiplying (IV) by $3v$, and subtracting from (III,) we have $4v^3 - 39v = -35$. Multiplying by v, we have $4v^4 - 39v^2 = -35v$
Dividing by 4, $v^4 - \frac{39}{4}v^2 = -\frac{35}{4}v$; add v^2 to each side, and $v^4 - \frac{35}{4}v^2 = v^2 - \frac{35}{4}v \therefore v^4 - v^2 = \frac{35}{4}(v^2 - v) \therefore (v^2 - v)(v^2 + v - \frac{35}{4}) = 0$
$\therefore v^2 - v = 0$ (V), and $v^2 + v = \frac{35}{4}$ (VI)
From (V), $v = 0$ or 1; and from (VI), $v = \frac{5}{2}$ or $-\frac{7}{2}$. But $2v^2 + 2z^2 = 13$
$\therefore z = \frac{1}{2}\sqrt{26}$; or $\pm\frac{1}{2}\sqrt{22}$; or $\pm\frac{1}{2}$; or $\pm\frac{1}{2}\sqrt{-23}$
Then $x = v + z = 1 \pm \frac{1}{2}\sqrt{22}$; or $\frac{5}{2} \pm \frac{1}{2}$; or $-\frac{7}{2} \pm \sqrt{-23}$
$y = v - z =$ values as obtained above.

NOTE.—The values $v = 0$, and $z = \frac{1}{2}\sqrt{26}$ are derived from the v, by which we multiplied equation (IV).

35. By Algebra Article 106, we have $\sqrt{y^2 + 1} = \dfrac{\sqrt{x + 9}}{3}$; whence $x = 9y^2$; substituting this in the 2nd equation, we get $9y^2(y^2 + 2y + 1) = 36y^3 + 64$; or $9y^4 - 18y^3 + 9y^2 = 64$; or $9y^2(y^2 - 2y + 1) = 64 \therefore 3y(y - 1) = \pm 8$; or $y^2 - y = \pm\frac{8}{3}$, whence $y = \frac{1}{6}(3 \pm \sqrt{105})$, or $\frac{1}{6}(3 \pm \sqrt{-87})$
And $x = 9y^2 = \frac{3}{2}(19 \pm \sqrt{105})$; or $\frac{3}{2}(-13 \pm \sqrt{-87})$

36. Multiplying the 2nd equation by x, we have $x^4 + xy^3 = x$, but $x^4 + y^4 = x \therefore x^4 + y^4 = x^4 + xy^3 \therefore y^4 - xy^3 = 0$; that is $y^3(y - x) = 0$ whence $y^3 = 0 \therefore y = 0$, and hence $x = 1$. Also $y - x = 0 \therefore y = x$; whence $2x^3 = 1$, and $y = x = \frac{1}{2}\sqrt[3]{4}$

37. Dividing the 1st equation by x^3, we have $\left(x^3+\frac{1}{x^3}\right)y=y^2+1$ $\therefore x^3+\frac{1}{x^3}=y+\frac{1}{y}$ (I). Again dividing the 2nd equation by y^3, we have $\left(y^3+\frac{1}{y^3}\right)x=9(x^2+1)$ $\therefore y^3+\frac{1}{y^3}=9\left(x+\frac{1}{x}\right)$ or $\frac{1}{3}\left(y^3+\frac{1}{y^3}\right)=3\left(x+\frac{1}{x}\right)$ (II)

Now adding equations (I) and (II) together, we have $\frac{1}{3}\left(y^3+\frac{1}{y^3}\right)+y+\frac{1}{y}=x^3+3x+\frac{3}{x}+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)^3$ $\therefore y^3+\frac{1}{y^3}+3y+\frac{3}{y}=\left(y+\frac{1}{y}\right)^3=3\left(x+\frac{1}{x}\right)^3$, or extracting the cube root of each, $y+\frac{1}{y}=\left(x+\frac{1}{x}\right)\sqrt[3]{3}$. But (I) $y+\frac{1}{y}=x^3+\frac{1}{x^3}$ $\therefore x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)\sqrt[3]{3}$, and factoring $\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)=\left(x+\frac{1}{x}\right)\sqrt[3]{3}$ $\therefore x+\frac{1}{x}=0$, or $x^2=-1$, or $x=\pm\sqrt{-1}$. Also $x^2-1+\frac{1}{x^2}=\sqrt[3]{3}$; or $x^2+2+\frac{1}{x^2}=3+\sqrt[3]{3}$ (III) by adding 3 to each side, $\therefore x+\frac{1}{x}=\pm\sqrt{3+\sqrt[3]{3}}$, similarly by taking 1 from each side of (III) and then taking $\sqrt{\ }$; $x-\frac{1}{x}=\pm\sqrt{\sqrt[3]{3}-1}$ Then by addition, we have $x=\pm\frac{1}{2}\{\sqrt{3+\sqrt[3]{3}}+\sqrt{\sqrt[3]{3}-1}\}$ And $y+\frac{1}{y}=\left(x+\frac{1}{x}\right)\sqrt[3]{3}=\pm\sqrt[3]{3}\times\sqrt{\sqrt[3]{3}+3}=\sqrt{3+3\sqrt[3]{9}}$ $\therefore y^2+2+\frac{1}{y^2}=3+3\sqrt[3]{9}$, or taking 4 from each $y^2-2+\frac{1}{y^2}=3\sqrt[3]{9}-1$ $\therefore y-\frac{1}{y}=\pm\sqrt{3\sqrt[3]{9}-1}$. Hence by addition $y=\pm\frac{1}{2}\{\sqrt{3+3\sqrt[3]{9}}+\sqrt{3\sqrt[3]{9}-1}\}$. Also since $x+\frac{1}{x}=0$; $\left(x+\frac{1}{x}\right)\sqrt[3]{3}=0$ $\therefore y+\frac{1}{y}=0$, whence $y=\pm\sqrt{-1}$

38. By transposition $\frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{x}{y} + \frac{y}{x} = \frac{27}{4}$, or adding 2 to each side, $\left(\frac{x^2}{y^2} + 2 + \frac{y^2}{x^2}\right) + \left(\frac{x}{y} + \frac{y}{x}\right) = \frac{35}{4}$; completing the square $\left(\frac{x}{y} + \frac{y}{x}\right)^2 + \left(\frac{x}{y} + \frac{y}{x}\right) + \frac{1}{4} = 9 \therefore \frac{x}{y} + \frac{y}{x} + \frac{1}{2} = \pm 3$ $\therefore \frac{x}{y} + \frac{y}{x} = \frac{5}{2}$ or $-\frac{7}{2} \therefore x^2 + y^2 = \frac{5xy}{2}$ or $-\frac{7xy}{2}$

Again by squaring the 2nd equation, we get $x^2 + y^2 = 2xy + 4$ $\therefore 2xy + 4 = \frac{5xy}{2}$ or $-\frac{7xy}{2}$; whence $xy = 8$ or $-\frac{8}{11}$. Then since $x^2 - 2xy + y^2 = 4$, and $4xy = 32$ or $-\frac{32}{11}$; we have by addition $x^2 + 2xy + y^2 = 36$ or $\frac{12}{11}$ $\therefore x + y = \pm 6$ or $\pm \frac{2}{11}\sqrt{33}$, and $x - y = 2$ $\therefore x = 4$ or -2; or $1 \pm \frac{1}{11}\sqrt{33}$; and similarly $y = 2$, or -4, or $-1 \pm \frac{1}{11}\sqrt{33}$

39. From the 1st equat., we get $(\sqrt{x} + \sqrt{y}) + \sqrt{5}(\sqrt{x} + \sqrt{y})^{\frac{1}{2}} = 10$, completing the square $(\sqrt{x} + \sqrt{y}) + \sqrt{5}(\sqrt{x} + \sqrt{y})^{\frac{1}{2}} + \frac{5}{4} = \frac{45}{4}$ $\therefore (\sqrt{x} + \sqrt{y})^{\frac{1}{2}} + \frac{1}{2}\sqrt{5} = \pm \frac{3}{2}\sqrt{5}$; whence $(\sqrt{x} + \sqrt{y})^{\frac{1}{2}} = \sqrt{5}$ or $-2\sqrt{5}$ $\therefore \sqrt{x} + \sqrt{y} = 5$ or 20 (II). Taking the former of these values, and raising to the 5th power, we have

$$x^{\frac{5}{2}} + 5x^2y^{\frac{1}{2}} + 10x^{\frac{3}{2}}y + 10xy^{\frac{3}{2}} + 5x^{\frac{1}{2}}y^2 + y^{\frac{5}{2}} = 3125$$

But $x^{\frac{5}{2}} + y^{\frac{5}{2}} = 275$

$$\therefore 5x^2y^{\frac{1}{2}} + 10x^{\frac{3}{2}}y + 10xy^{\frac{3}{2}} + 5x^{\frac{1}{2}}y^2 = 2850$$

$$x^{\frac{1}{2}}y^{\frac{1}{2}}\left(x^{\frac{3}{2}} + 2xy^{\frac{1}{2}} + 2x^{\frac{1}{2}}y + y^{\frac{3}{2}}\right) = 570 \text{ (III)}$$

But cubing equation (I), and multiplying it by $x^{\frac{1}{2}}y^{\frac{1}{2}}$, we have

$$x^{\frac{1}{2}}y^{\frac{1}{2}}\left(x^{\frac{3}{2}} + 3xy^{\frac{1}{2}} + 3x^{\frac{1}{2}}y + y^{\frac{3}{2}}\right) = 125x^{\frac{1}{2}}y^{\frac{1}{2}} \text{ (IV)}$$

Subtracting (III) from (IV), we have

$x^{\frac{1}{2}}y^{\frac{1}{2}}\left(xy^{\frac{1}{2}} + x^{\frac{1}{2}}y\right)$; that is $xy\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right) = 125x^{\frac{1}{2}}y^{\frac{1}{2}} - 570$

But $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 5 \therefore 5xy = 125x^{\frac{1}{2}}y^{\frac{1}{2}} - 570$

Hence $xy - 25x^{\frac{1}{2}}y^{\frac{1}{2}} = -114 \therefore x^{\frac{1}{2}}y^{\frac{1}{2}} = 19$ or 6

Then $x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 25$; and $4x^{\frac{1}{2}}y^{\frac{1}{2}} = 24$ or 76

$\therefore x - 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 1$ or -51; or $x^{\frac{1}{2}} - y^{\frac{1}{2}} = \pm 1$ or $\pm\sqrt{-51}$; and $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 5 \therefore x^{\frac{1}{2}} = 3$, or 2, or $\frac{1}{2}(5 \pm \sqrt{-51})$; whence $x = 9$, or 4, or $\frac{1}{2}(-13 \pm \sqrt{-51})$; similarly $y = 4$ or 9, or $\frac{1}{2}(-13 \mp \sqrt{-51})$ By using throughout the value $\sqrt{x} + \sqrt{y} = 20$, other values of x and y may be similarly found.

40. From the 1st equation $(x + y)(x^2 - xy + y^2) = x - y$ $\therefore x^2 - xy + y^2 = \dfrac{x - y}{x + y}$, and from the 2nd equat. $x^2 - axy + y^2 = 0$ $\therefore$ by subtraction, we have $(a - 1)xy = \dfrac{x - y}{x + y} = \sqrt{\dfrac{x^2 - 2xy + y^2}{x^2 + 2xy + y^2}}$ $= \sqrt{\dfrac{axy - 2xy}{axy + 2xy}} = \sqrt{\dfrac{(a - 2)xy}{(a + 2)xy}} = \sqrt{\dfrac{a - 2}{a + 2}} \therefore xy = \dfrac{\sqrt{\dfrac{a-2}{a+2}}}{a - 1} = b$, suppose. Then $y = \dfrac{b}{x}$, and $y^2 = \dfrac{b^2}{x^2}$; substituting these values in the 2nd equation, we have $x^2 + \dfrac{b^2}{x^2} = ab$; or $x^4 - abx^2 = -b^2$; whence $x^2 = \dfrac{b}{2}(a \pm \sqrt{a^2 - 4})$; and therefore $x = \pm\sqrt{\dfrac{b}{2}(a \pm \sqrt{a^2 - 4})}$ $= \pm \frac{1}{2}\sqrt{b}\{2(a \pm \sqrt{a^2 - 4})\}^{\frac{1}{2}} = \pm \frac{1}{2}\sqrt{b}(\sqrt{a + 2} \pm \sqrt{a - 2})$ by Algebra Art. 189. And $y = \dfrac{b}{x} = \pm 2\sqrt{b}(\sqrt{a + 2} \mp \sqrt{a - 2})$. See Algebra Art. 181

41. From the 1st equation $xy + ax - ay - a^2$; that is $(x - a)(y + a) = 0 \therefore x - a = 0$; or $x = a$. Also $y + a = 0 \therefore y = -a$ From the 2nd equat., substituting $x = a$, we have $a + y^2 + a^3 = 0$ $\therefore y = \pm a^{\frac{1}{2}}\sqrt{-(a^2 + 1)}$. Again substituting $-a$ for y in the 2nd equation, we have $x + a^2 + a^3 = 0$; whence $x = -a^2(a + 1)$

42. Squaring the first equation, we have $x^4 + y^4 + a^4 + 2x^2(y^2 + a^2) + 2a^2y^2 = 0$; and subtracting this fron the 2nd equat., we get $x^2(y^2 - a^2) - 2a^2y^2 = 0 \therefore x^2(y^2 - a^2) = 2a^2y^2$ (1). From the 1st given equat. $x^2 = -(y^2 + a^2) \therefore x^2(y^2 - a^2) = -(y^4 - a^4)$ $\therefore -(y^4 - a^4) = 2a^2y^2$, or $y^4 + 2a^2y^2 = a^4$; whence $y^2 = a^2(-1 \pm \sqrt{2})$ and $y = \pm a\sqrt{-1 \pm \sqrt{2}}$. Also $x^2 = -(y^2 + a^2) = \mp a^2\sqrt{2}$ $\therefore x = \sqrt{\mp a^2\sqrt{2}}$

43. From 2nd equat. $x^6 - 3y^3 + a^6 + 3x^4y - x^2y^2 - a^3x^4 - 2a^3x^2 = 0$ (I), and 1st equation $\times (x^4 - y^2)$ gives $x^6 - 3y^3 + 3x^4y - x^2y^2 + a^3x^4 - a^3y^2 = 0$ (II); then (I) − (II) gives $a^6 - 2a^3x^4 - 2a^3x^2 + a^3y^2 = 0$ or dividing by $-a^3$, we have $2x^4 + 2x^2 - a^3 - y^2 = 0$ (III). But from first of the given equations $y^2 = \left\{-\dfrac{(x^2+a^3)}{3}\right\}^2 = \left(\dfrac{x^2+a^3}{3}\right)^2$

$\therefore$ substituting this in (III), $2x^4 + 2x^2 - \left(\dfrac{x^2+a^3}{3}\right)^2 - a^3 = 0$; that is when clear of fractions $18x^4 + 18x^2 - x^4 - 2a^3x^2 - a^6 - 9a^3 = 0$, that is $17x^4 + 2(9 - a^3)x^2 = a^3(9 + a^3)$

whence $x^2 = \dfrac{a^3 - 9 \pm 3\sqrt{9 - 15a^3 + 2a^6}}{17}$, and

$\therefore x = \pm \frac{1}{17}\{17(a^3 - 9 \pm 3\sqrt{9 - 15a^3 + 2a^6})\}^{\frac{1}{2}}$

And $y = -\dfrac{x^2 + a^3}{3} = -\frac{1}{17}(6a^3 - 3 \pm \sqrt{9 - 15a^3 + 2a^6})$

44. Raising the 1st equation to the 4th power, we have $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 = x^4 + y^4 - 2x^2y^2 - 4xy(x^2 - 2xy + y^2) = a^4$ But $x^2 - 2xy + y^2 = a^2$, and $x^4 + y^4 = b^4$ $\therefore b^4 - 2x^2y^2 - 4a^2xy = a^4$; that is $2x^2y^2 + 4a^2xy = b^4 - a^4$, whence $xy = \dfrac{-2a^2 \pm \sqrt{2a^4 + 2b^4}}{2}$

Then $x^2 - 2xy + y^2 = a^2$, and multiplying value of xy by 4, and adding, we have $x^2 + 2xy + y^2 = -3a^2 \pm 2\sqrt{2a^4 + 2b^4} = m^2$, suppose $\therefore x - y = a$, and $x + y = m$; whence $x = \dfrac{m+a}{2}$, and $y = \dfrac{m-a}{2}$ where $m = \pm\sqrt{2\sqrt{2a^4 + 2b^4} - 3a^2}$

45. From 1st equation $x^2 + y^2 = a^2 + xy$, and squaring this, we have $x^4 + 2x^2y^2 + y^4 = a^4 + 2a^2xy + x^2y^2$ $\therefore x^4 - x^2y^2 + y^4 - 2a^2xy = a^4$ subtracting the 2nd given equation from this, we have $2x^2y^2 - 2a^2xy = a^4 - b^4$; whence $xy = \dfrac{a^2 \pm \sqrt{3a^4 - 2b^4}}{2} = c^2$, suppose Then $xy = c^2$, and $x^2 - xy + y^2 = a^2$ $\therefore x^2 - 2xy + y^2 = a^2 - c^2$, and $x - y = \pm\sqrt{a^2 - c^2}$. Also $x^2 + 2xy + y^2 = a^2 + 3c^2$; whence $x + y = \pm\sqrt{a^2 + 3c^2}$ $\therefore x = \pm\frac{1}{2}(\sqrt{a^2 + 3c^2} \pm \sqrt{a^2 - c^2})$, and $y = \pm\frac{1}{2}(\sqrt{a^2 + 3c^2} \mp \sqrt{a^2 - c^2})$, where $c^2 = \dfrac{a^2 \pm \sqrt{3a^4 - 2b^4}}{2}$

46. From the 2nd equation, we have $x^4 - 2x^2y^2 + 2x^2(a-1) + y^4 - 2y^2(a-1) + (a-1)^2 = a^2 - 4a + 4$ $\therefore$ extracting the square root of each, we have $x^2 - y^2 + (a-1) = \pm(a-2)$. To find the values of x and y which are independent of a use $x^2 - y^2 + (a-1) = +(a-2)$. Then $x^2 - y^2 + 1 = 0$, or $x^2 = y^2 - 1$; $x^2 - 3 = y^2 - 4$ (III). Again from the first given equation $3x^6 - 18x^4 + 27x^2 = 3x^2(x^2-3)^2 = 2y^6 - 11y^4 + 52y^2 + 27$ But (III) $x^2 - 3 = y^2 - 4$, and $x^2 = y^2 - 1$ $\therefore$ $3(y^2-1)(y^2-4)^2 = 2y^6 - 11y^4 + 52y^2 + 27$; that is $3y^6 - 27y^4 + 72y^2 - 48 = 2y^6 - 11y^4 + 52y^2 + 27$ $\therefore$ $y^6 - 16y^4 + 20y^2 - 75 = 0$; multiplying by -4, we have $-4y^6 + 64y^4 - 80y^2 + 300 = 0$, and adding to each side $y^8 - 20y^4 + 100$ we get $y^8 - 4y^6 + 44y^2 - 80y^2 + 400 = y^8 - 20y^4 + 100$. Then taking the square root $y^4 - 2y^2 + 20 = \pm(y^4 - 10)$; that is $y^2 = 15$, or $y^4 - y^2 = -5$, whence $y^2 = 15$, or $\frac{1}{2}(1 \pm \sqrt{-19})$, and $x^2 = y^2 - 1 = 14$, or $\frac{1}{2}(\pm\sqrt{-19} - 1)$ whence $y = \pm\sqrt{15}$, or $\pm\sqrt{\frac{1}{2}(1 \pm \sqrt{-19})} = \&c.$

47. From the 1st given equat. $y^4 - 2x^2y^2 + x^4 + 4(y^2 - x^2) + 5 = 2\sqrt{4(y^2-x^2)^3 + 5(y^2-x^2)^2}$; that is $(y^4 - 2x^2y^2 + x^4) - 2(y^2-x^2)\sqrt{4(y^2-x^2)+5} + \{4(y^2-x^2)+5\} = 0$ $\therefore$ extracting the square root, we have $y^2 - x^2 - \sqrt{4(y^2-x^2)+5} = 0$; or $y^2 - x^2 = \sqrt{4(y^2-x^2)+5}$ $\therefore$ $(y^2-x^2)^2 - 4(y^2-x^2) = 5$ whence $y^2 - x^2 = 5$ or -1, taking $y^2 - x^2 = 5$, we have from the 2nd given equation $y^4 - 3y^2 + 1 = 5x^2 - 8x + 8x\sqrt{x^2 - 2x + 5} + 4$ $= 5x^2 - 8x + 4x\sqrt{4x^2 - 8x + 20} + 4 = 5x^2 - 8x + 4x\sqrt{3y^2 + x^2 - 8x + 5} + 4$ since $15 = 3y^2 - 3x^2$. Hence by transposition, we have $y^4 = 3y^2 + x^2 - 8x + 5 + 4x\sqrt{3y^2 + x^2 - 8x + 5} + 4x^2$, and taking the square root $y^2 = \pm(\sqrt{3y^2 + x^2 - 8x + 5} + 2x)$; using the positive sign, $y^2 - 2x = \sqrt{3y^2 + x^2 - 8x + 5}$; squaring $y^4 - 4xy^2 + 4x^2 = 3y^2 + x^2 - 8x + 5$ $\therefore$ $y^4 - 4xy^2 + 4x^2 - 4y^2 + 8x + 4 = x^2 - y^2 + 9$; but $y^2 - x^2 = 5$ $\therefore$ $y^4 - 4x^2y^2 + 4x^2 - 4y^2 + 8x + 4 = 9 - 5 = 4$ $\therefore$ $y^2 - 2x - 2 = \pm 2$ $\therefore$ $y^2 - 2x = 4$ or 0; but $y^2 = x^2 + 5$ $\therefore$ $x^2 - 2x + 5 = 4$, or $x^2 - 2x + 5 = 0$; whence $x = 1$, or $1 \pm 2\sqrt{-1}$ and $y = \pm\sqrt{x^2 + 5} = \pm\sqrt{6}$, or $\pm\sqrt{2 \pm 4\sqrt{-1}}$

48. From the 2nd given equation, we have $x^2y^2 - 6xy\sqrt{y^2 - x^2} + 9(y^2 - x^2) = 16(y^2 - x^2)$ $\therefore$ extracting the square root, we have $xy - 3\sqrt{y^2 - x^2} = \pm 4\sqrt{y^2 - x^2}$ $\therefore xy = 7\sqrt{y^2 - x^2}$ or $-\sqrt{y^2 - x^2}$, and $\therefore x^2y^2 = 49(y^2 - x^2)$ or $(y^2 - x^2)$ From the 1st given equation $x^4 - y^4 - 4x^2 + 4y^2 = 4x^2 - 12$ $\therefore x^4 - 8x^2 + 16 = y^4 - 4y^2 + 4$; whence $x^2 - 4 = \pm (y^2 - 2)$ that is $y^2 = x^2 - 2$, or $6 - x^2$ $\therefore y^2 - x^2 = -2$; $x^2y^2 = 49(y^2 - x^2)$ or $(y^2 - x^2)$; that is $x^2(x^2 - 2) = 49(-2)$ or -2; that is $x^4 - 2x^2 = -98$ or -2; whence $x^2 = 1 \pm \sqrt{-97}$ or $1 \pm \sqrt{-2}$ and $y^2 = x^2 - 2 = -1 \pm \sqrt{-97}$, or $-1 \pm \sqrt{-2}$. Also since $y^2 = 6 - x^2$, we have by substitution $x^2(6 - x^2) = 49(6 - 2x^2)$ or $6 - 2x^2$; that is $x^4 - 6x^2 = 98x^2 - 294$, or $= 2x^2 - 6$ that is $x^4 - 104x^2 = -294$, whence $x^2 = 52 \pm \sqrt{2410}$; or $x^4 - 8x^2 = -6$, whence $x^2 = 4 \pm \sqrt{10}$. And $y^2 = 6 - x^2 = -46 \mp \sqrt{2410}$ or $2 \mp \sqrt{10}$; whence $x =$ &c.

Exercise LV.

1. Let $x =$ one part, then $19 - x =$ other, and $x(19 - x) = 84$ $x^2 - 19x = -84$, $x^2 - 19x + (\frac{19}{2})^2 = \dfrac{361 - 336}{4} = \frac{25}{4}$ $\therefore x - \frac{19}{2} = \pm \frac{5}{2}$ $x = 12$ or 7; whence the numbers are 12 and 7

2. Let $x =$ greater, then $17 - x =$ less; $x - (17 - x) = 2x - 17$ $=$ difference; then $x(2x - 17) = 30$, $2x^2 - 17x = 30$, whence $x = 10$, and the numbers are 10 and 7

3. Let $x =$ length, then $x - 12 =$ breadth, and $x(x - 12) = 2080$ that is $x^2 - 12x = 2080$, whence $x = 52$ and sides are 52 and 40 rods.

4. Let $x =$ greater, then $x - 9 =$ the less, and $x^2 + (x - 9)^2 = 353$ that is $2x^2 - 18x + 81 = 353$; $x^2 - 9x = 136$, whence $x = 17$ or -8 and the numbers are 17 and 8, or -8 and -17

5. Let x = one part, then $16 - x$ = other, then $x(16 - x) + x^2 + (16 - x)^2 = 208$; that is $x^2 - 16x = -48$, whence $x = 12$ or 4, and the numbers are 12 and 4

6. Let x = gain per cent. = buying price of wheat; then $\frac{x}{100}$ = gain per dollar on buying price, and $x \times \frac{x}{100} = \frac{x^2}{100}$ = gain on x dollars, i.e. gain on whole transaction; but $171 - x$ = whole gain, whence $\frac{x^2}{100} = 171 - x$; or $x^2 + 100x = 17100$, whence $x = \$90$, buying price of wheat.

Proof. $\$81 + \$90 = \$171$; also if he gain \$81 on \$90, he gains at the rate of \$9 on \$10, or \$90 on \$100.

7. Let x = number of sheep bought, then $\frac{80}{x}$ = price in dollars; then $\frac{80}{x+4} = \frac{80}{x} - 1$; that is $x^2 + 4x = 320$, whence $x = 16$.

8. Let x, y and z be the digits, z being the right hand one; then $x^2 + y^2 + z^2 = 104$ (I); $y^2 = 2xz + 4$ (II); $100x + 10y + z - 594 = 100z + 10y + x$ (III). Substituting (II) in (I), we have $x^2 + 2xz + z^2 = 100$; whence $x + z = 10$ (IV). Reducing (III), we have $99x - 99z = 594$, or $x - z = 6$ (V). Adding (IV) and (V) together, we have $2x = 16$; whence $x = 8$. Also $x + z = 10$ $\therefore z = 2$, and $y^2 = 2xy + 4 = 32 + 4 = 36$; whence $y = 6$. Hence the required number is 862

9. Let x = number of sheep bought, then $\frac{240}{x}$ = price per sheep, $x - 15$ = number sold, and $\frac{240}{x} + \frac{2}{5}$ = selling price; then $(x - 15)\left(\frac{240}{x} + \frac{2}{5}\right) = 216$, that is $\left(\frac{600 + x}{5x}\right)(x - 15) = 108$; or $x^2 + 45x = 9000$; whence $x = 75$, and $\frac{240}{75} = \$3{\cdot}20$. Hence number bought was 75, and price per sheep \$3·20

10. Let x = one number, then $10 - x$ = other, and $x^3 + (10 - x)^3$, that is $x^3 + 1000 - 300x + 30x^2 - x^3 = 280$, that is $30x^2 - 300x$

$= -720$; or $x^2 - 10x = -24$, whence $x = 6$ or 4, and the required numbers are 6 and 4

11. Let x = less, then $24 - x$ = greater, and $x(24 - x) = 35(24 - x - x)$, that is $24x - x^2 = 35(24 - 2x) = 840 - 70x$; or $x^2 - 94x = -840$, whence $x = 10$ or 84, and $24 - x = 14$ or -60 $\therefore$ the required parts of 24 are 10 and 14, or 84 and -60

12. Let x and y be the numbers, then $x + y = xy = x^2 - y^2$; $x + y = x^2 - y^2$, whence dividing by $x + y$, we have $x - y = 1$; or $x = 1 + y$. Also $x + y = xy$, that is $1 + y + y = y(1 + y)$; or $1 + 2y = y + y^2$ $\therefore$ $y^2 - y = 1$; whence $y = \frac{1}{2}(1 \pm \sqrt{5})$, and $x = 1 + y = 1 + \frac{1}{2}(1 \pm \sqrt{5}) = \frac{1}{2}(3 \pm \sqrt{5})$

13. Let x = circumference of hind wheel, and y = that of fore wheel in yards; then $\frac{120}{x}$ and $\frac{120}{y}$ = revolutions made by each in going 120 yards. Also by second condition $\frac{120}{x+1}$ and $\frac{120}{y+1}$ = revolutions made in 120 yards.

$$\text{Then } \left.\begin{aligned} \frac{120}{x} &= \frac{120}{y} - 6 \\ \frac{120}{x+1} &= \frac{120}{y+1} - 4 \end{aligned}\right\}; \text{ or } \left.\begin{aligned} \frac{20}{x} &= \frac{20}{y} - 1 \\ \frac{30}{x+1} &= \frac{30}{y+1} - 1 \end{aligned}\right\} \begin{aligned} &(\text{I}) \text{ by dividing by } 6 \\ &(\text{II}) \quad \text{"} \quad 4 \end{aligned}$$

$\therefore$ $20x - 20y = xy$ (III) and $29x - 31y = xy + 1$ (IV). Subtracting (III) from (IV), we have $9x - 11y = 1$; or $9x = 11y + 1$ $\therefore$ $x = \frac{11y+1}{9}$

Substituting this in (III), we have $\frac{20(11y + 1)}{9} - 20y = y\left(\frac{11y+1}{9}\right)$

that is $220y + 20 - 180y = 11y^2 + y$ $\therefore$ $11y^2 - 39y = 20$; whence $y = 4$, and $x = \frac{11y + 1}{9} = \frac{45}{9} = 5$. Hence circumferences of wheels are 4 and 5 yards respectively.

14. Let x = one fraction, then $\frac{29}{15} - x$ = other, and $\frac{1}{x} + \frac{1}{\frac{29}{15} - x}$ = sum of their reciprocals $\therefore$ $\frac{1}{x} + \frac{15}{29 - 15x} = \frac{29}{12}$; that is $12(29 - 15x) + 180x = 29x(29 - 15x)$, whence by reduction

$435x^2 - 841x = -348$; or $x^2 - \frac{841}{435} = -\frac{348}{435}$; $x^2 - \frac{841}{435}x + (\frac{841}{870})^2 - \frac{707281}{756900} - \frac{605520}{756900} = \frac{101761}{756900}$, whence $x = \frac{841}{870} \pm \frac{319}{870} = \frac{1160}{870}$ or $\frac{522}{870}$ $= \frac{4}{3}$ or $\frac{3}{5}$, and $\frac{29}{15} - x = \frac{29}{15} - \frac{20}{15} = \frac{9}{15} = \frac{3}{5}$; or $\frac{29}{15} - \frac{9}{15} = \frac{20}{15} = \frac{4}{3}$. Hence fractions are $\frac{4}{3}$ and $\frac{3}{5}$

15. Let x = number of children, then $\dfrac{46800}{x}$ = share of each; then $\dfrac{46800}{x-2} = \dfrac{46800}{x} + 1950$, that is $\dfrac{24}{x-2} = \dfrac{24}{x} + 1$; whence $x^2 - 2x = 48$ $\therefore x = 8$ = number of children.

16. Let x = number of hours the clock is too fast, then, since the shadow on the dial moves from 1 to 5, the clock will strike the hours from $2 + x$ to $5 + x$ inclusive; i. e. will strike $2 + x + 3 + x + 4 + x + 5 + x = 14 + 4x$ strokes, and last stroke will be $5 + x$. Then $(5 + x)^2 - 41$ = number of minutes the clock is too fast above the x hours; i. e. $25 + 10x + x^2 - 41$; i. e. $x^2 + 10x - 16$. But hours too fast + minutes too fast = whole number of strokes; that is $x + x^2 + 10x - 16 = 14 + 4x$, whence $x^2 + 7x = 30$ $\therefore x = 3$ or 10, and $x^2 + 10x - 16 = 9 + 30 - 16 = 23$ $\therefore$ the clock is 3 h. 23 m. too fast. The second answer 10 is excluded by the limitation that the clock does not strike 12 during the time.

17. Let x = hours travelled by each = miles per hour travelled by slower, then $x + 3$ = miles per hour travelled by faster; $x^2 + x(x + 3) = 2x^2 + 3x = 324$, whence $x = 12$. Hence slower travelled $12 \times 12 = 144$ miles, and the faster $12 \times 15 = 180$ miles.

18. Let x = number, then $\dfrac{144}{x}$ = share of each $\therefore \dfrac{144}{x+2} + 1 = \dfrac{144}{x}$ whence $x^2 + 2x = 288$ $\therefore x = 16$ = number at first.

19. Let x = left hand, and y = right hand digit, then $\dfrac{10x + y}{xy} = 2$ (I), and $10x + y + 27 = 10y + x$ (II). From (I) $10x + y = 2xy$ (III) From (II) $9x - 9y = -27$, whence $x - y = -3$, or $x = y - 3$; substituting this in (III), we have $10(y - 3) + y = 2y(y - 3)$ $\therefore 2y^2 - 17y = -30$, whence $y = 6$, and $x = y - 3 = 3$. Hence the required number is 36

20. Let x = price of coffee, and y = price of sugar per lb. in cts.; then $60x + 80y = 2500$ (I). Also $\frac{800}{y}$ = lbs. of sugar for \$8, and $\frac{1000}{x}$ = lbs. of coffee for \$10; then $\frac{800}{y} = \frac{1000}{x} + 24$ (II) From (II) by reduction $100x - 125y = 3xy$ (III). From (I) $x = \frac{125-4y}{3}$, substituting this for x in (III), we have $\frac{100(125-4y)}{3} - 125y = 3y \times \frac{(125-4y)}{3}$, whence by reduction $6y^2 - 575y = -6250$ $\therefore y = 12\frac{1}{2}$ cents, and $x = \frac{125-4y}{3} = 25$ cents.

21. Let x and y = number of days required by B and C respectively to finish the work; then in 1 day A does $\frac{1}{18}$th; $B, \frac{1}{x}$th; and $C, \frac{1}{y}$th of the field; $\frac{1}{18} + \frac{1}{x} : \frac{1}{x} :: \$36 : \frac{36}{x} \div \frac{x+18}{18x} = \frac{648}{x+18}$ = what B would have received, had C not been called in; but B worked 10 days $\therefore$ he did receive $\frac{10}{x} \times 36 = \frac{360}{x}$ Then $\left(\frac{648}{x+18} - \frac{360}{x}\right)$dollars $= \$1{\cdot}50 = \$\frac{3}{2}$; whence by reduction $x^2 - 174x = -4320$ $\therefore x^2 - 174x + (87)^2 = 7569 - 4320 = 3249$ $\therefore x = 30$ = days B would require. And $\frac{10}{18} + \frac{10}{30} + \frac{4}{y} = 1$ $\therefore \frac{4}{y} = 1 - (\frac{10}{18} + \frac{10}{30}) = 1 - \frac{16}{18} = \frac{2}{18}$ $\therefore 2y = 72$, and $y = 36$ = days C would require tocradle the field.

PROOF.—If C had not been called in, they would have taken $11\frac{1}{4}$ days to finish the work, and A's share would have been $\$2 \times 11\frac{1}{4} = \$22{\cdot}50$. Hence B's share would have been \$13·50, but since, when C is called in, B only works 10 days, he receives only $\frac{10}{30} = \frac{1}{3}$ of $\$36 = \$12 = \$1{\cdot}50$ less than he would have otherwise received.

22. Let x and y = the number of feet in the side of the base; then $5xy - 4xy - xy = 80 + x + y$ (I); also $\sqrt{25 + x^2 + y^2} = \frac{3}{40}(x^2 + y^2)$ (II). From (II), we get $3(x^2 + y^2) = 40\sqrt{25 + x^2 + y^2}$

That is $(x^2 + y^2 + 25) - \frac{40}{3}\sqrt{x^2 + y^2 + 25} = 25$

$(x^2 + y^2 + 25) - \frac{40}{3}(x^2 + y^2 + 25)^{\frac{1}{2}} + \frac{400}{9} = \frac{400}{9} + \frac{225}{9} = \frac{625}{9}$

$\therefore \sqrt{x^2 + y^2 + 25} - \frac{20}{3} = \pm \frac{25}{3} \therefore \sqrt{x^2 + y^2 + 25} = 15$ or $-\frac{5}{3}$; squaring, we have $x^2 + y^2 + 25 = 225$ or $\frac{25}{9}$ $\therefore$ taking the former value, $x^2 + y^2 = 200$ (III), but by (I) $2xy = 160 + 2(x + y)$ (IV). Adding (IV) and (III), we have $x^2 + 2xy + y^2 = 360 + 2(x + y)$. Hence $(x + y)^2 - 2(x + y) = 360 \therefore (x + y)^2 - 2(x + y) + 1 = 361$; $(x + y) - 1 = \pm 19$, $x + y = 20$ or -18 $\therefore$ again taking the former value, $x + y = 20$, and hence $xy = 80 + 20 = 100$

$$\left.\begin{array}{rl} x^2 + 2xy + y^2 &= 400 \\ 4xy &= 400 \\ \hline x^2 - 2xy + y^2 &= 0 \\ \therefore x - y &= 0 \end{array}\right\} \begin{array}{l} \therefore x = y \therefore 2x = 20, \text{ and } x = 10 = y \\ \\ \therefore \text{the base is a square whose side is 10 ft.} \end{array}$$

23. Let x = distance B has travelled when he meets A, then $x + 15$ = distance A has travelled; Also since A has yet to travel x miles, and accomplishes it in 2 hours, his rate of travelling is $\frac{x}{2}$ miles per hour; also B's rate is $\frac{x+15}{4\frac{1}{2}} = \frac{2x+30}{9}$

Then time A travels before they meet $= \frac{x + 15}{\frac{1}{2}x} = \frac{2x + 30}{x}$

time B travels before they meet $= \dfrac{x}{\frac{2x + 30}{9}} = \frac{9x}{2x + 30}$;

$\therefore \frac{2x+30}{x} = \frac{9x}{2x+30}$, that is $x^2 - 24x = 180$; whence $x = 30$ = rate

Hence distance $= x + x + 15 = 75$ miles. B's rate $= \frac{2x+30}{9} = \frac{90}{9}$ $= 10$ miles per hour. A's rate $= \frac{x}{2} = 15$ miles per hour.

24. Let x and xy be the two numbers, the latter being the greater; then $x^2y = x^2y^2 - x^2$, whence $y^2 - y = 1$, and $y = \frac{1}{2}(1 \pm \sqrt{5})$. Also $x^2y^2 + x^2 = x^3y^3 - x^3 \therefore y^2 + 1 = xy^3 - x$; whence $x = \frac{y^2+1}{y^3-1} = \frac{\frac{1}{2}(3 \pm \sqrt{5}) + 1}{(2 \pm \sqrt{5}) - 1} = \frac{\frac{1}{2}(5 \pm \sqrt{5})}{1 \pm \sqrt{5}} = \frac{\frac{1}{2}\sqrt{5}(\sqrt{5} \pm 1)}{1 \pm \sqrt{5}} = \pm \frac{1}{2}\sqrt{5}$

25. Let x and y = hours required by Bacchus and Silenus respectively; then Bacchus would drink $\frac{1}{x}$ of it in 1 hour, hence in y hours he would drink $\frac{y}{x}$ths of it, and in $\frac{2}{3}y$ hours Bacchus would drink $\frac{2y}{3x}$ths of the cask full $\therefore 1 - \frac{2y}{3x}$ = part drunk by Silenus, and since he drinks $\frac{1}{y}$ of the cask in 1 hour, the time he required to drink part remaining, was $\left(1 - \frac{2y}{3x}\right) \div \frac{1}{y}$ $= y - \frac{2y^2}{3x}$. Had both drunk together, Bacchus would only have consumed $\frac{1}{2}\left(1 - \frac{2y}{3x}\right) = \frac{1}{2} - \frac{y}{3x}$, and Silenus would have taken $\frac{1}{2} + \frac{y}{3x}$; hence when drinking together, time taken by Bacchus was $\left(\frac{1}{2} - \frac{y}{3x}\right) \div \frac{1}{x} = \frac{x}{2} - \frac{y}{3}$, and the time taken by Silenus was $\left(\frac{1}{2} + \frac{y}{3x}\right) \div \frac{1}{y} = \frac{y}{2} + \frac{y^2}{3x}$; hence $\frac{x}{2} - \frac{y}{3} = \frac{y}{2} + \frac{y^2}{3x}$ (I)

Also $\frac{2y}{3} + \left(y - \frac{2y^2}{3x}\right)$ = time taken when drinking separately $= \frac{x}{2} - \frac{y}{3} + 2$ (II). From (I) $3x^2 - 5yx = 2y^2$. From (II) $12xy - 4y^2 = 3x^2 + 12x$ (III) $x^2 - \frac{5y}{3}x + \frac{25y^2}{36} = \frac{25y^2}{9} + \frac{2y^2}{3}$ $= \frac{49y^2}{36}$; $x - \frac{5y}{6} = \pm\frac{7y}{6}$. Hence $x = \frac{12y}{6} = 2y$; substituting this value of x in (III), we have $24y^2 - 4y^2 = 3(2y)^2 + 12(2y)$, that is $24y^2 - 4y^2 = 12y^2 + 24y$ $\therefore 8y^2 - 24y = 0$; $y^2 = 3y$; $y = 3$, whence $x = 6$

Exercise LVI.

1. $\frac{a}{b} \times \frac{c}{a^2} \times \frac{ab}{cd} = \frac{1}{d} = 1 : d$

2. $\frac{a^2-b^2}{a^3+b^3} \times \frac{(a-b)^2}{a} \times \frac{a^2-ab+b^2}{(a-b)^3} = \frac{(a-b)(a+b)(a-b)^2(a^2-ab+b^2)}{a(a+b)(a^2-ab+b^2)(a-b)^3}$
$= \frac{1}{a} = 1 : a$

3. $\frac{(x-5)(x+3)}{(x-5)(x+2)} \times \frac{(x+2)(x-1)}{(x+3)(x+5)} \times \frac{(x+7)(x+5)}{(x-1)(x+1)} = \frac{x+7}{x+1}$
$= x+7 : x+1$

4. $\frac{a^3+b^3}{a^2+b^2} \gtrless \frac{a^2+b^2}{a+b}$, according as $a^4 + a^3b + ab^3 + b^4 \gtrless a^4 + 2a^2b^2 + b^4$; or as $a^3b + ab^3 \gtrless 2a^2b^2$; or as $a^2 + b^2 \gtrless 2ab$; but $a^2 + b^2$ is greater than $2ab$ (Algebra Art. 134, Note 2)
$\therefore \frac{a^3+b^3}{a^2+b^2} > \frac{a^2+b^2}{a+b}$

5. $\frac{x^2+y^2}{x^2-y^2} \gtrless \frac{(x+y)^4}{x^4-x^3y+x^2y^2-xy^3+y^4}$, according as $x^6 - x^5y + 2x^4y^2 - 2x^3y^3 + 2x^2y^4 - xy^5 + y^6 \gtrless x^6 + 4x^5y + 5x^4y^2 - 5x^2y^4 - 4xy^5 - y^6$; or as $7x^2y^4 + 3xy^5 + 2y^6 \gtrless 5x^5y + 2x^3y^3 + 3x^4y^2$; or as $7x^2y^3 + 3xy^4 + 2y^5 \gtrless 5x^5 + 2x^3y^2 + 3x^4y$; or as $y^3(7x^2 + 3xy + 2y^2) \gtrless x^3(5x^2 + 3xy + 2y^2)$
Now since $x\sqrt[3]{5} > y\sqrt[3]{7}$, cubing we have $5x^3 > 7y^3$ $\therefore x^3 > y^3$
$\therefore y^3(3xy+2y^2) < x^3(3xy+2y^2)$; also $7x^2y^3 < 5x^5$ $\because 7y^3 < 5x^3$
$\therefore y^3(7x^2+3xy+2y^2) < x^3(5x^2+3xy+2y^2)$
$\therefore \frac{x^2+y^2}{x^2-y^2} < \frac{(x+y)^4}{x^4-x^3y+x^2y^2-xy^3+y^4}$

6. Let x = the quantity to be subtracted from each term; then $\frac{a-x}{b-x} = \frac{c}{d}$ $\therefore ad - dx = bc - cx$; $cx - dx = bc - ad$ $\therefore x = \frac{bc-ad}{c-d}$

7. Let x = the quantity to be added to each term; then $\frac{m+x}{n+x} = 1$, whence $m + x = n + x$; $x - x = m - n$; $x(1-1) = m - n$;

$$x = \frac{m - n}{1 - 1} = \frac{m - n}{0} = \infty$$

8. $$\frac{(a+b)}{(a-b)} \times \left(\frac{a^2}{b^2} - \frac{a^2}{a^2}\right) \times \frac{b^3}{(a+b)^3} = \frac{(a+b)}{(a-b)} \times \left(\frac{a^2}{b^2} - 1\right) \times \frac{b^3}{(a+b)^3}$$

$$= \frac{(a+b)}{(a-b)} \times \frac{a^2 - b^2}{b^2} \times \frac{b^3}{(a+b)^3} = \frac{b}{a+b} = b : a + b$$

9. Since $a : c :: c : b$, $c = \sqrt{ab}$; then $\frac{(a+c)^2}{(b+c)^2} = \frac{a^2 + 2ac + c^2}{b^2 + 2bc + c^2}$

$$= \frac{a^2 + 2a\sqrt{ab} + ab}{b^2 + 2b\sqrt{ab} + ab} = \frac{a(a + 2\sqrt{ab} + b)}{b(b + 2\sqrt{ab} + a)} = \frac{a}{b} = a : b$$

10. $\frac{a^2 - b^2}{a^2 + b^2} \gtrless \frac{a-b}{a+b}$, according as $a^3 + a^2b - ab^2 - b^3 \gtrless a^3 - a^2b + ab^2 - b^3$; or as $a^2b - ab^2 \gtrless ab^2 - a^2b$; or as $2a^2b \gtrless 2ab^2$; or as $a \gtrless b$

Exercise LVII.

1. Let x = the quantity to be added; then $a + x : b + x :: c + x : d + x$ $\therefore ad + ax + dx + x^2 = bc + bx + cx + x^2$; $ax + dx - bx - cx = bc - ad$

$$\therefore x = \frac{bc - ad}{a - b - c + d}$$

2. If $a : b :: c : d$; then $\frac{a}{b} = \frac{c}{d}$. If it be possible, let x be a quantity added to each, so that $a + x : b + x :: c + x : d + x$; then $\frac{a+x}{b+x} = \frac{c+x}{d+x}$, whence as above $x = \frac{bc - ad}{a - b - c + d}$; but since $\frac{a}{b} = \frac{c}{d}$, we have $bc = ad$ $\therefore bc - ad = 0$ $\therefore x = \frac{0}{a - b - c + d} = 0$

3. $\frac{a}{b} = \frac{c}{d} \therefore \frac{a^2}{b^2} = \frac{c^2}{d^2}$; also $\frac{m}{n} = \frac{p}{q} \therefore \frac{m}{2n} = \frac{p}{2q}$. Multiplying equals by equals $\frac{ma^2}{2nb^2} = \frac{pc^2}{2qd^2}$; then Algebra Art. 106 $\frac{ma^2 - 2nb^2}{ma^2 + 2nb^2} = \frac{pc^2 - 2qd^2}{pc^2 + 2qd^2} \therefore ma^2 - 2nb^2 : pc^2 - 2qd^2 :: ma^2 + 2nb^2 : pc^2 + 2qd^2$

4. Let x = one number, then $\frac{24}{x}$ = other

And $x^3 - \left(\frac{24}{x}\right)^3 : \left(x - \frac{24}{x}\right)^3 :: 19 : 1$

Hence $x^3 - \left(\frac{24}{x}\right)^3 = 19\left\{x^3 - 72x + \frac{1728}{x} - \left(\frac{24}{x}\right)^3\right\}$

$$x^3 - \left(\frac{24}{x}\right)^3 = 19x^3 - 1368x + \frac{32832}{x} - 19\left(\frac{24}{x}\right)^3$$

$$18\left(\frac{24}{x}\right)^3 - 18x^3 = \frac{32832}{x} - 1368x$$

$$\left(\frac{24}{x}\right)^3 - x^3 = \frac{1824}{x} - 76x;\ \text{or}\ \left(\frac{24}{x}\right)^3 - x^3 = 76\left(\frac{24}{x} - x\right)$$

Dividing each side by $\frac{24}{x} - x$, we have $\left(\frac{24}{x}\right)^2 + 24 + x^2 = 76$, that is $\frac{576}{x^2} + x^2 = 52$; $x^4 - 52x^2 = -576$; $x^4 - 52x + (26)^2 = 100$; $x^2 = 26 \pm 10 = 36$ or $16 \therefore x = \pm 6$ or ± 4, and the numbers are ± 6 and ± 4

5. Let x = one part, then $20 - x$ = other part; then $x : 20 - x :: 9 : 1 \therefore x = 180 - 9x$; or $x = 18$, and $20 - x = 2$ Let y be the mean proportional between these; then $18 : y :: y : 2$; or $y^2 = 36 \therefore y = 6$

6. $\frac{x}{y} = \frac{a^3}{b^3}$; also $\frac{a}{b} = \frac{\sqrt[3]{c+x}}{\sqrt[3]{d+y}} \therefore \frac{a^3}{b^3} = \frac{c+x}{d+y} \therefore \frac{x}{y} = \frac{c+x}{d+y}$ $\therefore dx + xy = cy + xy$, or $dx = cy$

7. Dividing the equation by $(a+b-c-d)(a-b-c+d)$, we have $\frac{a+b+c+d}{a+b-c-d} = \frac{a-b+c-d}{a-b-c+d}$. Then Art. 106, we have

$\frac{2a+2b}{2c+2d} = \frac{2a-2b}{2c-2d}$ ∴ Art. 106, $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ ∴ Art. 106, $\frac{2a}{2b} = \frac{2c}{2d}$;

or $\frac{a}{b} = \frac{c}{d}$ ∴ $a : b :: c : d$

8. Let x and y = the numbers; then $x+y : s :: x-y : d :: xy : p$ ∴ $p(x+y) = sxy$, and $p(x-y) = dxy$ ∴ $\frac{s}{p} = \frac{x+y}{xy}$, and $\frac{d}{p} = \frac{x-y}{xy}$

$$\left.\begin{array}{l} \frac{s}{p} = \frac{1}{y} + \frac{1}{x} \\ \frac{d}{p} = \frac{1}{y} - \frac{1}{x} \end{array}\right\} \begin{array}{l} \text{By addition } \frac{s+d}{p} = \frac{2}{y}, \text{ whence } y = \frac{2p}{s+d} \\ \text{By subtraction } \frac{2}{x} = \frac{s-d}{p}, \text{ whence } x = \frac{2p}{s-d} \end{array}$$

9. Let x = speed in yards of faster train per second, and y = speed of slower; then in $2''$ the former passes over $2x$, and the latter $2y$ yards, consequently $2x + 2y$ = length of the faster train; also $30x - 30y$ = length of faster train, ∴ $30x - 30y = 2x + 2y$, or $28x = 32y$, or $7x = 8y$, ∴ $x : y :: 8 : 7$

10. Let x = *A*'s money, and y = *B*'s; then $x + 150 : y - 50 :: 3 : 2$, whence $2x + 300 = 3y - 150$; or $2x - 3y = -450$ (I). Also $x - 50 : y + 100 :: 5 : 9$, whence $9x - 450 = 5y + 500$; or $9x - 5y = 950$ (II). Multiplying (I) by 9, and (II) by 2, we have

$$\left.\begin{array}{l} 18x - 27y = -4050 \text{ (III)} \\ 18x - 10y = 1900 \text{ (IV)} \end{array}\right\} \text{Subtracting (III) from (IV) } 17y = 5950,$$

whence y = \$350 = *B*'s stock; $2x - 3y = 2x - 1050 = -450$, whence $2x = 600$, and x = \$300 = *A*'s stock.

11. $b = \sqrt{ac}$ ∴ $b^2 = ac$; $2b^2 = 2ac$; $b^2 = 2ac - b^2$ ∴ adding $a^2 + c^2$ to each, $a^2 + b^2 + c^2 = a^2 + 2ac + c^2 - b^2$; or $a^2 + b^2 + c^2$ $= (a+c)^2 - b^2 = (a+c-b)(a+c+b)$ ∴ $1 = \frac{(a+c-b)(a+c+b)}{a^2+b^2+c^2}$

or $\frac{1}{a+c+b} = \frac{a+c-b}{a^2+b^2+c^2}$; or $\frac{a+c+b}{(a+c+b)^2} = \frac{a-b+c}{a^2+b^2+c^2}$

∴ $a+b+c : (a+b+c)^2 :: a-b+c : a^2+b^2+c^2$

12. $a : c :: a : c$, $\therefore$ multiplying each term of the latter ratio by $a - c$, we have $a : c :: a(a-c) : c(a-c)$
$\therefore a : c :: a(\sqrt{a}-\sqrt{c})(\sqrt{a}+\sqrt{c}) : c(\sqrt{a}-\sqrt{c})(\sqrt{a}+\sqrt{c})$
$\therefore a : c :: \sqrt{a}(\sqrt{a}-\sqrt{c})(\sqrt{a}+\sqrt{c})\sqrt{a} : \sqrt{c}(\sqrt{a}-\sqrt{c})(\sqrt{a}+\sqrt{c})\sqrt{c}$
$\therefore a : c :: (a-\sqrt{ac})(a+\sqrt{ac}) : (\sqrt{ac}-c)(\sqrt{ac}+c)$; but $\sqrt{ac} = b$
$\therefore a : c :: (a-b)(a+b) : (b-c)(b+c)$ since by hypothesis $\sqrt{ac} = b$

13. Let x = the number; then $x+3 : x+8 :: x+8 : x+17$ $\therefore (x+3)(x+17) = (x+8)^2$, that is $x^2+20x+51 = x^2+16x+64$; or $4x = 13$ $\therefore x = 3\frac{1}{4}$

14. Let D and d = diameters of a sovereign and shilling respectively, and t and T = thickness of a sovereign and shilling respectively; then $md = nD$, and $pt = qT$, and since circles are to one another as the squares of their diameters, we have quantity of metal in sovereign : quantity of metal in shilling $:: D^2T : d^2t$, or $:: \frac{D^2}{d^2} : \frac{t}{T}$

But $md = nD$ $\therefore D : d :: m : n$ $\therefore \frac{D}{d} = \frac{m}{n}$, similarly $\frac{t}{T} = \frac{q}{p}$

$\therefore$ quantity of gold in sov. : quan. silver in shil. $:: \frac{m^2}{n^2} : \frac{q}{p}$

But a sovereign = 20 s., $\therefore$ quantity of gold in bulk equal to a shil. : quantity of silver in a shii. $:: \frac{20q}{p} : \frac{m^2}{n^2}$, or $:: 20n^2q : m^2p$

15. $\frac{a}{b} = \frac{c}{d}$ $\therefore$ multiplying both by $\frac{\cdot\dot{4}\dot{2}}{11\frac{1}{7}}$, we have $\frac{\cdot\dot{4}\dot{2}a}{11\frac{1}{7}b} = \frac{\cdot\dot{4}\dot{2}c}{11\frac{1}{7}d}$

$\therefore$ Art. 106, $\frac{\cdot\dot{4}\dot{2}a+11\frac{1}{7}b}{11\frac{1}{7}b} = \frac{\cdot\dot{4}\dot{2}c+11\frac{1}{7}d}{11\frac{1}{7}d}$, or multiplying by $11\frac{1}{7}$

$\frac{\cdot\dot{4}\dot{2}a+11\frac{1}{7}b}{b} = \frac{\cdot\dot{4}\dot{2}c+11\frac{1}{7}d}{d}$ (I); also $\frac{a}{b} = \frac{c}{d}$ $\therefore \frac{4a}{5b} = \frac{4c}{5d}$ $\therefore \frac{4a-5b}{5b} = \frac{4c-5d}{5d}$; or $\frac{4a-5b}{b} = \frac{4c-5d}{d}$ (II). Dividing (I) by (II), we have

$\frac{\cdot\dot{4}\dot{2}a+11\frac{1}{7}b}{b} \times \frac{b}{4a-5b} = \frac{\cdot\dot{4}\dot{2}c+11\frac{1}{7}d}{d} \times \frac{d}{4c-5d}$; that is $\frac{\cdot\dot{4}\dot{2}a+11\frac{1}{7}b}{4a-5b} = \frac{\cdot\dot{4}\dot{2}a+11\frac{1}{7}d}{4c-5d}$

16. If $a:b::b:c::c:d$, $ac = b^2$ and $bd = c^2$ $\therefore d = \frac{c^2}{b} = \frac{c^2}{\sqrt{ac}} = \frac{\sqrt{c^3}}{\sqrt{a}}$

$(a+b)(c-d) = (a+\sqrt{ac})\left(c - \frac{\sqrt{c^3}}{\sqrt{a}}\right) = \sqrt{a}(\sqrt{a}+\sqrt{c})(\sqrt{a}-\sqrt{c})\frac{c}{\sqrt{a}}$

$= c^2(a-c)$; $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$ $\therefore \frac{a^3}{b^3} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{d}$ $\therefore \frac{a}{b} = \frac{\sqrt[3]{a}}{\sqrt[3]{d}}$

$\therefore a : \sqrt[3]{a} :: b : \sqrt[3]{d}$.

Exercise LVIII.

1. $mx^2 + ny \propto cx^2 - dy$ $\therefore mx^2 + ny = p(cx^2 - dy) = pcx^2 - pdy$ $\therefore pcx^2 - mx^2 = ny + pdy$; that is $x^2(pc - m) = (n + pd)y$ $\therefore x^2 = \left(\frac{n+pd}{pc-m}\right)y$ $\therefore x = \left(\sqrt{\frac{n+pd}{pc-m}}\right)\sqrt{y}$; But since n, p, m, d and c are all constant, $\sqrt{\left(\frac{n+pd}{pc-m}\right)}$ is constant $\therefore x \propto \sqrt{y}$

2. $x = my$; $7 = 3m$ $\therefore m = \frac{7}{3}$ $\therefore x = \frac{7}{3}y$

3. $x = p + \frac{m}{y}$; then $1 = p + \frac{m}{3}$ (I), and $2 = p + m$ (II) Subtracting (I) from (II), $1 = \frac{2}{3}m$ $\therefore m = \frac{3}{2}$, and $p + m = p + \frac{3}{2} = 2$ $\therefore p = \frac{1}{2}$ $\therefore x = p + \frac{m}{y} = \frac{1}{2} + \frac{\frac{3}{2}}{y} = \frac{1}{2} + \frac{3}{2y} = \frac{1}{2} + \frac{3}{30} = \frac{1}{2} + \frac{1}{10} = \frac{1}{5}$

4. $x^2 = my^3$; $4 = 64m$ $\therefore m = \frac{1}{16}$; $x^2 = \frac{1}{16}y^3$ $\therefore x = \frac{1}{4}y\sqrt{y}$

5. $x = m + nxy$ $\therefore 2 = m + 6n$ (I), and $3 = m - 9n$ (II) Subtracting (I) from (II), we have $1 = -15n$ $\therefore n = -\frac{1}{15}$ $m - \frac{6}{15} = 2$ $\therefore m = 2 + \frac{6}{15} = \frac{12}{5}$. Then $x - nxy = m$; $x(1 - ny) = m$ $\therefore x = \frac{m}{1-ny} = \frac{\frac{12}{5}}{1-(-\frac{1}{15}y)} = \frac{\frac{12}{5}}{\frac{15+y}{15}} = \frac{36}{15+y}$

6. $y = m + nx + px^2$; then $0 = m + 3n + 9p$ (I) $-12 = m + 5n + 25p$ (II), and $-32 = m + 7n + 49p$ (III)

$$\left.\begin{array}{ll} \text{Subtracting (I) from (II), we get} & -12 = 2n + 16p \quad \text{(IV)} \\ \text{" \quad (I) from (III), "} & -32 = 4n + 40p \quad \text{(V)} \end{array}\right\}$$

Dividing (IV) by 2, and (V) by 4, we have $-6 = n + 8p$ (VI)

$-8 = n + 10p$ (VII)

Subtracting (VI) from (VII), we have $-2 = 2p \therefore p = -1$; $-6 = n - 8 \therefore n = 2$; $0 = m + 6 - 9 \therefore m = 3 \therefore y = 3 + 2x - x^2$

7. $y = mx^2 + \dfrac{n}{x}$; then $7 = 25m + \dfrac{n}{5}$ (I); $5 = 81m + \dfrac{n}{9}$ (II)

Dividing (II) by 5, and (I) by 9, we have $\frac{7}{9} = \dfrac{25n}{9} + \dfrac{n}{45}$ (III), and $1 = \dfrac{81m}{5} + \dfrac{n}{45}$ (IV). Subtracting (III) from (IV) $\dfrac{2}{9} = \dfrac{604m}{45}$ $\therefore m = \frac{5}{302} \therefore n = 35 - 125m = 32\frac{281}{302}$; then $y = mx^2 + \dfrac{n}{x} = \dfrac{5x^2}{302} + \dfrac{9945}{302x}$

8. $y = mb^2 + mx^2 \therefore \dfrac{a^2}{b} = mb^2 + ma^2 - mb^2 = ma^2 \therefore \dfrac{1}{b} = m$

$y = \dfrac{b^2}{b} + \dfrac{x^2}{b} = b + \dfrac{x^2}{b}$

9. $z - x - y = m$, and $(x + y + z)(x - y - z) = nyz$; that is $\{x + (y + z)\}\{x - (y + z)\}$ or $x^2 - (y + z)^2 = nyz$. Adding $4yz$ to each side, $x^2 - (y - z)^2 = (n + 4)yz$; that is $(x - y + z)(x + y - z) = (n + 4)yz$; but $z - x - y = m \therefore x + y - z = -m \therefore -m(x - y + z) = (n + 4)yz \therefore x - y + z = -\dfrac{n + 4}{m}yz \therefore x - y + z \propto yz$

10. Let x^2 = number of cars attached, then decrease of speed $\propto x^2$, and is $\therefore = mx^2$; then $24 - mx$ = speed of train, $\therefore 20 = 24 - 2m \therefore 2m = 4$, or $m = 2$; then $24 - 2x^2$ = speed when x^2 waggons are attached. Now if speed is reduced to 0, we have $24 - 2x^2 = 0 \therefore x = 12$, and $\therefore x^2 = 144$ = number of cars required to completely stop the train, $\therefore$ greatest number it can move = 143

Exercise LIX.

1. $S_{31} = \{126 + (31 - 1)2\}\frac{31}{2} = \frac{126 + 60}{2} \times 31 = 2883$

$S_n = \{126 + (n - 1)2\}\frac{n}{2} = (126 + 2n - 2)\frac{n}{2} = \frac{124 + 2n}{2} \times n = n(62 + n)$

2. $S_{22} = \{-400 + (22 - 1) \times 12\}\frac{22}{2} = (-400 + 252)11 = -1628$

$S_n = \{-400 + (n - 1) \times 12\}\frac{n}{2} = \frac{(-400 + 12n - 12)}{2} \times n = n(6n - 206)$

3. $S_{17} = \{4 + (17 - 1)\frac{3}{2}\}\frac{17}{2} = (4 + 24)\frac{17}{2} = 238$

$S_{2m+p} = \{4 + (2m + p - 1)\frac{3}{2}\}\frac{2m + p}{2} = \{\frac{5}{2} + (2m + p)\frac{3}{2}\}\frac{2m + p}{2}$
$= \frac{5}{4}(2m + p) + \frac{3}{4}(2m + p)^2$

4. $S_{11} = (\frac{4}{3} + (11 - 1) \times -\frac{2}{3}\}\frac{11}{2} = (\frac{4}{3} - \frac{20}{3})\frac{11}{2} = -\frac{16}{3} \times \frac{11}{2} = -\frac{176}{6} = -29\frac{1}{3}$

5. $17^{\text{th}} = 2 + (17 - 1)3 = 2 + 16 \times 3 = 2 + 48 = 50$

$28^{\text{th}} = 2 + (28 - 1)3 = 2 + 27 \times 3 = 2 + 81 = 83$

$n^{\text{th}} = 2 + (n - 1)3 = 2 + 3n - 3 = 3n - 1$

6. $17^{\text{th}} = 3 + (17 - 1) \times -5 = 3 + 16 \times -5 = 3 - 80 = -77$

$28^{\text{th}} = 3 + (28 - 1) \times -5 = 3 + 27 \times -5 = 3 - 135 = -132$

$n^{\text{th}} = 3 + (n - 1) \times -5 = 3 - 5n + 5 = 8 - 5n$

7. $17^{\text{th}} = 2\frac{1}{2} + (17 - 1)\frac{5}{7} = \frac{5}{2} + 16 \times \frac{5}{7} = \frac{5}{2} + \frac{80}{7} = \frac{195}{14} = 13\frac{13}{14}$

$28^{\text{th}} = 2\frac{1}{2} + (28 - 1)\frac{5}{7} = \frac{5}{2} + 27 \times \frac{5}{7} = \frac{5}{2} + \frac{135}{7} = 21\frac{11}{14}$

$n^{\text{th}} = 2\frac{1}{2} + (n - 1)\frac{5}{7} = \frac{5}{2} + \frac{5}{7}n - \frac{5}{7} = \frac{5}{2} - \frac{5}{7} + \frac{5}{7}n = \frac{25}{14} + \frac{5}{7}n = \frac{5}{14}(5 + 2n)$

8. $d = \frac{33 - 3}{5 - 1} = \frac{30}{4} = 7\frac{1}{2}$; hence series $= 3 + 10\frac{1}{2} + 18 + 25\frac{1}{2} + 33$

9. $d = \frac{-66 - 9}{6 - 1} = -\frac{75}{5} = -15$; hence series $= 9 - 6 - 21 - 36 - 51 - 66$

10. $d = \frac{100 - (-1)}{9 - 1} = \frac{100 + 1}{8} = \frac{101}{8} = 12\frac{5}{8}$; hence series $= -1 + 11\frac{5}{8} + 24\frac{1}{4} + 36\frac{7}{8} + 49\frac{1}{2} + 62\frac{1}{8} + 74\frac{1}{4} + 87\frac{3}{8} + 100$

11. $S_{73} = \{2 + (73 - 1)1\}\frac{73}{2} = (2 + 72)\frac{73}{2} = 37 \times 73 = 2701$

12. n^{th} term $= 1 + (n - 1)2 = 1 + 2n - 2 = 2n - 1$

13. $S_n = \{2 + (n-1)2\}\frac{n}{2} = (2 + 2n - 2)\frac{n}{2} = 2n \times \frac{n}{2} = n^2$

14. $S_t = \{2a + (t-1)2a\}\frac{t}{2} = (2a + 2at - 2a)\frac{t}{2} = 2at \times \frac{t}{2} = at^2$

15. 20th term $= a + (20-1)2a = a + 19 \times 2a = a + 38a = 39a$

tth term $= a + (t-1)2a = a + 2at - 2a = 2at - a = a(2t-1)$

16. Let $x - 3y$, $x - y$, $x + y$, $x + 3y$ represent the numbers

Then $(x-3y)^2 + (x+3y)^2 = 2x^2 + 18y^2 = 200$

$(x - y)^2 + (x + y)^2 = 2x^2 + 2y^2 = 136$

$\therefore 16y^2 = 64$

Hence $y^2 = 4$ or $y = \pm 2$ $\therefore$ $2x^2 = 136 - 2y^2 = 136 - 8 = 128$; or $x^2 = 64$; or $x = \pm 8$ $\therefore$ the series is $\pm 14 \pm 10 \pm 6 \pm 2$

17. Let $x - 3y$, $x - y$, $x + y$, $x + 3y$ represent the numbers; then $(x-3y)(x-y)(x+y)(x+3y) = (x^2 - 9y^2)(x^2 - y^2)$ $= (x^2 - 36)(x^2 - 4)$*; $x^4 - 40x^2 + 144 = 1680$, or $x^4 - 40x^2 = 1536$; $x^4 - 40x^2 + 400 = 1936$ $\therefore$ $x^2 - 20 = \pm 44$ $\therefore$ $x^2 = +64$, or -24 Rejecting the latter value, we have $x^2 = 64$, or $x = \pm 8$; hence the series is $\pm 14 \pm 10 \pm 6 \pm 2$

18. Let $x - 2y$, $x - y$, x, $x + y$, $x + 2y$ represent the numbers, then $x - 2y + x - y + x + x + y + x + 2y = 5x = 25$ $\therefore$ $x = 5$ $(x - 2y)(x - y)(x + y)(x + 2y)x = (x^2 - 4y^2)(x^2 - y^2)x$ $= 5(25 - 4y^2)(25 - y^2) = 5(4y^4 - 125y^2 + 625) = 945$, or $4y^4 - 125y^2 = -436$; $y^4 - \frac{125}{4}y^2 + (\frac{125}{8})^2 = \frac{8649}{64}$ $\therefore$ $y^2 - \frac{125}{8} = \pm \frac{93}{8}$ whence, $y^2 = 4$ and $y = \pm 2$. Hence the series is 1, 3, 5, 7, 9 or 9, 7, 5, 3, 1

19. $S = (a + l)\frac{n}{2} = (60 + 1)\frac{60}{2} = 61 \times 30 = 1830$, i. e. since the principal on interest is \$60 the first day, and only \$1 the 60th day, the whole interest is equivalent to that of \$1830 for 1 day. Interest of \$60 for 360 days = \$3·60, or of \$1 for 360 days = \$0·06, or of \$1 for 1 day $= \frac{6}{360} = \frac{1}{60}$ of a cent; hence the interest of \$1 for

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* The common difference is given $= 4$ $\therefore$ $y = 2$.

1830 days, i. e. of \$1830 for 1 day = $\frac{1830}{60}$ cents, and since this is to be divided into 60 payments, each will be $\frac{1830}{60 \times 60} = \frac{61}{120}$ of a cent.

20. $S = \{2a + (n-1)d\}\frac{n}{2} = \{2 + (n-1)\}\frac{n}{2} = (n+1)\frac{n}{2} = \frac{n(n+1)}{2}$

21. Let $S = 1^2 + 2^2 + 3^2 + \ldots\ldots n^2$

Now $n^3 - (n-1)^3 = 3n^2 - 3n + 1$

$(n-1)^3 - (n-2)^3 = 3(n-1)^2 - 3(n-1) + 1$

$(n-2)^3 - (n-3)^3 = 3(n-2)^2 - 3(n-2) + 1$ Hence by addition

$n^3 = {}^*\{3(n-2)^2 + 3(n-1)^2 + 3n^2\} - \{3(n-2) + 3(n-1) + 3n\} + n$

$n^3 = 3(1^2 + 2^2 + \ldots\ldots n^2) - 3(1 + 2 + \ldots\ldots n) + n$

But by supposition, $1^2 + 2^2 + \ldots\ldots n^2 = S$, and it has been shown in question 20, that $1 + 2 + 3 \ldots\ldots n = \frac{n(n+1)}{2}$

Therefore $n^3 = 3S - \frac{3n(n+1)}{2} + n$

$$3S = n^3 + \frac{3n(n+1)}{2} - n = \frac{2n^3 + 3n(n+1) - 2n}{2}$$

$$3S = \frac{n(2n^2 + 3n + 3 - 2)}{2} = \frac{n(2n^2 + 3n + 1)}{2} = \frac{n(n+1)(2n+1)}{2}$$

$$\therefore S = \frac{n(n+1)(2n+1)}{2 \cdot 3} = \frac{n(n+1)(2n+1)}{6}$$

22. $S = \{2a + (n-1)d\}\frac{n}{2}$; $517 = \{4 + (n-1)9\}\frac{n}{2}$; $1034 = 4n + 9n(n-1)$; $1034 = 9n^2 - 5n$; $324n^2 - 180n + 25 = 37224 + 25 = 37249$; $18n - 5 = \pm 193$; $18n = 198$; $n = 11$

NOTE.—The negative value is inadmissible.

23. $l + l - d + l - 2d = 3l - 3d = 96$, or $l - d = 32$; $l - 3d + l - 4d + l - 5d + l - 6d = 4l - 18d = 86$, or $2l - 9d = 43$; $2l - 9d = 43$, and $2l - 2d = 64$ $\therefore$ $7d = 21$, and $d = 3$, whence $l = 35$;

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*NOTE.—The student must here read n as number, $n - 1$, one less than number, &c. Thus taking $n = 3$, then $n - 1 = 2$, $n - 2 = 1$; if n be taken as 4, we should have to take four addends as above.

and inverting the series 35, 32, 29, 26, &c , we have 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35

24. $l = 5$, and $l - 2d = 7$ $\therefore$ the sixth term or last term but one $= l - d = \dfrac{7+5}{2} = 6$, and the series is found by reversing the series 5, 6, 7, 8, 9, 10, 11

25. $S = bn + cn^2 = (b + cn)n = (2b + 2cn)\dfrac{n}{2} = (2b + 2c + 2cn - 2c)\dfrac{n}{2}$ $= \{2(b + c) + (n - 1)2c\}\dfrac{n}{2}$. But by formula $S = \{2a + (n - 1)d\}\dfrac{n}{2}$, whence it is evident that a, the first term of the series $= b + c$; d, the common difference $= 2c$. Then the t^{th} term $= a + (t - 1)d$ $= b + c + (t - 1)2c = b - c + 2ct$

26. The $(m - n)^{\text{th}}$ term $= a + (m - n - 1)d$;
the $(m + n)^{\text{th}}$ term $= a + (m + n - 1)d$ $\therefore$ the sum of the two terms $= 2a + (2m - 2)d = 2\{a + (m - 1)d\}$. Also the m^{th} term $= a + (m - 1)d$. Therefore, &c.

27. $$\left.\begin{array}{l}(p+q)^{\text{th}}\text{ term} = a + (p+q-1)d = m\\ (p-q)^{\text{th}}\text{ term} = a + (p-q-1)d = n\end{array}\right\}\therefore 2qd = m - n \therefore d = \frac{m-n}{2q}$$
But $a + (q - 1)d + pd = m$ $\therefore$ $a + (q - 1)d = m - pd = m - p \times \dfrac{m-n}{2q}$ $= m - (m - n)\dfrac{p}{2q}$; but $a + (q - 1)d = q^{\text{th}}$ term $\therefore$ q^{th} term $= m - (m - n)\dfrac{p}{2q}$

28. p^{th} term $= 7 - \dfrac{p}{2} = \frac{13}{2} - \dfrac{p}{2} + \frac{1}{2} = \frac{13}{2} - (p - 1)\frac{1}{2}$ $= \frac{13}{2} + (p - 1) \times -\frac{1}{2}$; but p^{th} term $= a + (p - 1)d$, whence $a = \frac{13}{2}$, and $d = -\frac{1}{2}$. Then sum of n terms $= S = \{2a + (n - 1)d\}\dfrac{n}{2}$ $= \{13 + (n - 1) \times -\frac{1}{2}\}\dfrac{n}{2} = (13 - \dfrac{n}{2} + \frac{1}{2})\dfrac{n}{2} = \left(\dfrac{27}{2} - \dfrac{n}{2}\right)\dfrac{n}{2}$ $= \dfrac{n}{4}(27 - n)$

29. Let $x-y$, x and $x+y$ = the numbers; then $(x-y)^2+x(x+y)$ $= x^2-2xy+y^2+x^2+xy = 2x^2-xy+y^2 = 16$, and $x^2+(x-y)(x+y)$ $= x^2+x^2-y^2 = 2x^2-y^2 = 14$. Subtracting the second from the first, $2y^2-xy=2$, or $x = \dfrac{2y^2-2}{y}$. Substituting this for x in the equation $2x^2-y^2=14$, we have $\dfrac{2(2y^2-2)^2}{y^2} - y^2 = 14$, or $7y^4-30y^2=-8$, $196y^4-840y^2+900 = -224+900 = 676$; $14y^2-30 = \pm 26$; $14y^2 = 56$ or 4, $y^2 = 4$ or $\frac{2}{7}$; rejecting this latter value we have $y = \pm 2$. Hence $x = \dfrac{8-2}{\pm 2} = \dfrac{6}{\pm 2} = \pm 3$, and the three numbers are 1, 3 and 5, or -5, -3 and -1

30. Let $x-3y$, $x-y$, $x+y$ and $x+3y$ represent the numbers; then $x-3y+x-y+x+y+x+3y = 4x = 20$ $\therefore$ $x=5$;
$\dfrac{1}{x-3y}+\dfrac{1}{x-y}+\dfrac{1}{x+y}+\dfrac{1}{x+3y} = \dfrac{4x^3-20xy^2}{x^4-10x^2y^2+9y^4} = \frac{25}{24}$
$\therefore$ $25(625-250y^2+9y^4) = 24(500-100y^2)$; or $9y^4-154y^2=-145$; $324y^4-5544y^2+23716 = -5220+23716 = 18496$ $\therefore$ $18y^2-154$ $= \pm 136$; $18y^2 = 290$ or 18 $\therefore$ $y^2 = 1$ or $\frac{145}{9}$, and $y = \pm 1$ or $\pm\frac{1}{3}\sqrt{145}$
Rejecting the latter value, we have 5 ∓ 3, 5 ∓ 1, 5 ± 1 and 5 ± 3; that is 2, 4, 6 and 8 or 8, 6, 4 and 2 for the series.

Exercise LX.

1. 6^{th} term $= 3\times 3^5 = 3\times 243 = 729$;
$$S_6 = \frac{3(3^6-1)}{3-1} = \frac{3\times(729-1)}{2} = 1092$$

2. 9^{th} term $= 1\times 2^8 = 1\times 256 = 256$;
$$S_9 = \frac{1(2^9-1)}{2-1} = \frac{512-1}{1} = 511$$

3. 7^{th} term $= \frac{2}{7}\times 2^6 = \frac{2}{7}\times 64 = \frac{128}{7} = 18\frac{2}{7}$;
$$S_7 = \frac{\frac{2}{7}(2^7-1)}{2-1} = \frac{2}{7}\times(128-1) = 36\frac{2}{7}$$

4. 12^{th} term $= 3 \times (-2)^{11} = 3 \times -2048 = -6144$;

$$S_{12} = \frac{3\{(-2)^{12} - 1\}}{-2-1} = \frac{3(4096-1)}{-3} = -4095$$

5. 6^{th} term $= 4 \times (-\frac{5}{4})^5 = 4 \times -\frac{3125}{1024} = -\frac{3125}{256} = -12\frac{53}{256}$;

$$S_6 = \frac{4\{(-\frac{5}{4})^6 - 1\}}{-\frac{5}{4}-1} = \frac{4(\frac{15625}{4096}-1)}{-\frac{9}{4}} = \frac{\frac{15625}{1024}-4}{-\frac{9}{4}} = \frac{\frac{15625-4096}{1024}}{-\frac{9}{4}}$$
$= -5\frac{1}{256}$

6. 8^{th} term $= 30 \times (-\frac{1}{2})^7 = 30 \times -\frac{1}{128} = -\frac{30}{128} = -\frac{15}{64}$;

$$S_8 = \frac{30\{1-(\frac{1}{2})^8\}}{1+\frac{1}{2}} = \frac{1275}{64} = 19\frac{59}{64}$$

7. $S_\infty = \dfrac{-1\frac{1}{3}}{1-(-\frac{2}{3})} = \dfrac{-\frac{4}{3}}{1+\frac{2}{3}} = \dfrac{-\frac{4}{3}}{\frac{5}{3}} = -\frac{4}{5}$

8. $S_\infty = \dfrac{\frac{2}{5}}{1-\frac{2}{3}} = \dfrac{\frac{2}{5}}{\frac{1}{3}} = \frac{6}{5} = 1\frac{1}{5}$

9. $S_\infty = \dfrac{7}{1-(-\frac{1}{2})} = \dfrac{7}{1+\frac{1}{2}} = \dfrac{7}{\frac{3}{2}} = \frac{14}{3} = 4\frac{2}{3}$

10. $S_\infty = \dfrac{64}{1-(-\frac{1}{2})} = \dfrac{64}{1+\frac{1}{2}} = \dfrac{64}{\frac{3}{2}} = \frac{128}{3} = 42\frac{2}{3}$

11. $S_\infty = \dfrac{\frac{623}{1000}}{1-\frac{1}{1000}} = \dfrac{\frac{623}{1000}}{\frac{1000-1}{1000}} = \dfrac{\frac{623}{1000}}{\frac{999}{1000}} = \frac{623}{999}$

12. $S_\infty = \dfrac{\frac{7}{10}}{1-\frac{1}{10}} = \dfrac{\frac{7}{10}}{\frac{9}{10}} = \frac{7}{9}$

13. $S_\infty = \frac{9}{10} + \dfrac{\frac{76}{1000}}{1-\frac{1}{100}} = \frac{9}{10} + \dfrac{\frac{76}{1000}}{\frac{99}{100}} = \frac{9}{10} + \frac{76}{990} = \frac{967}{990}$

14. $S_\infty = \frac{862}{1000} + \dfrac{\frac{32}{100000}}{1-\frac{1}{100}} = \frac{862}{1000} + \dfrac{\frac{32}{100000}}{\frac{99}{100}} = \frac{862}{1000} + \frac{32}{99000} = \frac{8537}{9900}$

15. $S_n = \dfrac{1(3^n-1)}{3-1} = \dfrac{3^n-1}{2} = \frac{1}{2}(3^n-1)$

16. $S_n = \dfrac{2\{(-\frac{2}{5})^n - 1\}}{-\frac{2}{5}-1} = \dfrac{2\{1-(-\frac{2}{5})^n\}}{\frac{7}{5}} = \frac{10}{7}\{1-(-\frac{2}{5})^n\}$

17. $S_{10} = \dfrac{2\{(\sqrt{2})^{10} - 1\}}{\sqrt{2}-1} = \dfrac{2(32-1)}{\sqrt{2}-1} = \dfrac{62}{\sqrt{2}-1} = \dfrac{62\sqrt{2}+62}{2-1}$
$= 62(1+\sqrt{2})$

18. $S_n = \dfrac{a^p\{(a^q)^n - 1\}}{a^q-1} = \dfrac{a^p(a^{qn}-1)}{a^q-1} = \dfrac{a^{qn+p}-a^p}{a^q-1}$

19. $r = \left(\frac{\frac{16}{81}}{1}\right)^{\frac{1}{5-1}} = (\frac{16}{81})^{\frac{1}{4}} = \frac{2}{3}$ $\therefore$ series $= 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81}$

20. $r = (\frac{13122}{2})^{\frac{1}{9-1}} = (6561)^{\frac{1}{8}} = 3$ $\therefore$ series $= 2 + 6 + 18 + 54 + 162 + 486 + 1458 + 4374 + 13122$

21. $r = \left(\frac{\frac{1}{9}}{9}\right)^{\frac{1}{5-1}} = (\frac{1}{81})^{\frac{1}{4}} = \frac{1}{3}$ $\therefore$ series $= 9 + 3 + 1 + \frac{1}{3} + \frac{1}{9}$

22. Let x, xy, xy^2 and xy^3 represent the four numbers; then $x + xy^2 = x(1 + y^2) = 148$, and $xy + xy^3 = xy(1 + y^2) = 888$ $\therefore 1 + y^2 = \frac{148}{x}$, and $1 + y^2 = \frac{888}{xy}$ $\therefore \frac{148}{x} = \frac{888}{xy}$ $\therefore 148y = 888$ $\therefore y = 6$; then $1 + y^2 = 1 + 36 = 37 = \frac{148}{x}$ $\therefore 37x = 148$ $\therefore x = 4$, and the series is 4, 24, 144 and 864

23. Let x, xy, xy^2 and xy^3 represent the numbers; then $x + xy = 15$ (I), and $xy^2 + xy^3 = y^2(x + xy) = 60$ (II). Dividing (II) by (I) we have $y^2 = 4$ $\therefore y = \pm 2$, and since $x(1 + y) = 15$, we have $x = \frac{15}{3}$, or $\frac{15}{-1} = 5$ or -15; hence the numbers are 5, 10, 20 and 40, or -15, 30, -60 and 120

24. Let xy^2, xy and x represent the number of dollars they severally had; then $xy^2 = x + 135$ (I), and $xy^2 + xy + x = 315$ (II) $\therefore xy + 2x = 180$ (III). From (I) $x = \frac{135}{y^2 - 1}$, and from (III) $x = \frac{180}{y + 2}$ $\therefore \frac{135}{y^2 - 1} = \frac{180}{y + 2}$ $\therefore \frac{3}{y^2 - 1} = \frac{4}{y + 2}$ $\therefore 4y^2 - 3y = 10$, whence $y = 2$ or $-\frac{5}{4}$; hence $x = \frac{180}{y + 2} = \frac{180}{4} = 45$, or $x = \frac{180}{2 - \frac{5}{4}} = \frac{180}{\frac{3}{4}} = 240$; hence the shares were \$180, \$90 and \$45.

Taking the negative value as above, gives us $x = \$240$, and the shares would be \$375, $-$ \$300 and \$240, which implies that the second receives \$300 less than nothing for his share, or in other words, instead of receiving anything he gives \$300 to be divided in addition to the \$315 among the other two.

25. Let x = the first number, and y = the common ratio of the 1st three numbers; then the numbers are x, xy, xy^2, xy^2+xy, and xy^2+2xy

And $xy+xy^2+(xy^2+xy)+(xy^2+2xy)=3xy^2+4xy=40$ (I); also $xy(xy^2+2xy)=x^2y^3+2x^2y^2=64$ (II). Multiplying (I) by xy and (II) by 3, and subtracting, we have $2x^2y^2=192-40xy$ $\therefore x^2y^2+20xy=96$; whence $xy=4$ or -24. From (I) $xy(3y+4)$ $=40 \therefore 3y+4=\dfrac{40}{xy}=\frac{40}{4}=10 \therefore 3y=6$, and $y=2 \therefore x=2$; hence the numbers are 2, 4, 8, 12 and 16

26. $$S = a+(a+b)r+(a+2b)r^2+\ldots\ldots\{a+(n-1)b\}r^{n-1}$$
$$Sr = ar+(a+b)r^2+\ldots\ldots\{a+(n-2)b\}r^{n-1}+\{a+(n-1)b\}r^n$$
$$S-Sr = a+br+br^2+br^3+\ldots\ldots br^{n-1}-\{a+(n-1)b\}r^n$$
$$S(1-r)=a+\frac{br(1-r^{n-1})}{1-r}-\{a+(n-1)b\}r^n$$
$$S=\frac{a}{1-r}+\frac{br(1-r^{n-1})}{(1-r)^2}-\frac{\{a+(n-1)b\}r^n}{1-r}$$
$$S=\frac{a-\{a+(n-1)b\}r^n}{1-r}+\frac{br(1-r^{n-1})}{(1-r)^2}$$

27. (I) $a^2+b^2+c^2-(a-b+c)^2=a^2+b^2+c^2-(a^2+b^2+c^2)$ $+2ab+2bc-2ac=2ab+2bc-2ac=2ab+2bc-2b^2$, (since $ac=b^2$) $=2b(a+c-b)$. Now $(a+c)^2-b^2=a^2+2ac+c^2-b^2$ $=a^2+2b^2+c^2-b^2=a^2+b^2+c^2$ a positive quantity $\therefore (a+c)^2>b^2$, and $\therefore a+c>b$, and $\therefore a+c-b$ is a positive quantity, and $\therefore 2b(a+c-b)$ is positive, $\therefore a^2+b^2+c^2-(a-b+c)^2$ is positive, $\therefore a^2+b^2+c^2>(a-b+c)^2$

(II) $(a+b+c+d)^2=(a+b)^2+(c+d)^2+2(a+b)(c+d)$; but $\dfrac{a}{b}=\dfrac{b}{c}=\dfrac{c}{d}\therefore\dfrac{a+b}{b}=\dfrac{b+c}{c}=\dfrac{c+d}{d}\therefore\dfrac{(a+b)(c+d)}{bd}=\dfrac{(b+c)^2}{c^2}$

$\therefore (a+b)(c+d)=\dfrac{bd}{c^2}(b+c)^2=(b+c)^2 \because bd=c^2 \therefore (a+b+c+d)^2$ $=(a+b)^2+(c+d)^2+2(b+c)^2$

28. (I) $(p+q)^{\text{th}}$ term $= ar^{p+q-1} = m$

$(p-q)^{\text{th}}$ term $= ar^{p-q-1} = n$

$\therefore \overline{a^2r^{2p-2}} = mn, \therefore ar^{p-1} = \sqrt{mn} = p^{\text{th}}$ term

(II) Also $\dfrac{ar^{p+q-1}}{ar^{p-q-1}} = r^{2q} = \dfrac{m}{n} \;\therefore\; r = \left(\dfrac{m}{n}\right)^{\frac{1}{2q}}$, and $\dfrac{1}{r^p} = \left(\dfrac{n}{m}\right)^{\frac{p}{2q}}$

$\therefore$ the q^{th} term $= ar^{q-1} = \dfrac{ar^{p+q-1}}{r^p} = \dfrac{m}{r^p} = m \times \dfrac{1}{r^p} = m\left(\dfrac{n}{m}\right)^{\frac{p}{2q}}$

29. Let x, xy and xy^2 represent the numbers; then $x + xy + xy^2 = 35$, and $xy : xy^2 - x :: 2 : 3 \;\therefore\; y : y^2 - 1 :: 2 : 3$, or $3y = 2y^2 - 2 \;\therefore\; 2y^2 - 3y = 2$, whence $y = 2$ or $-\frac{1}{2}$ $\therefore x + xy + xy^2 = x + 2x + 4x = 7x = 35 \;\therefore\; x = 5$; or $x + xy + xy^2 = x - \frac{1}{2}x + \frac{1}{4}x = \frac{3}{4}x = 35 \;\therefore\; x = \frac{140}{3} = 46\frac{2}{3}$, hence the numbers are 5, 10 and 20; or $46\frac{2}{3}$, $-23\frac{1}{3}$ and $11\frac{2}{3}$

30. Let x, xy and xy^2 represent the digits; then $100x + 10xy + xy^2$ = the number, and $x + xy + xy^2$ = sum of its digits; then $100x + 10xy + xy^2 : x + xy + xy^2 :: 124 : 7$, that is $100 + 10y + y^2 : 1 + y + y^2 :: 124 : 7$, whence $\dfrac{100 + 10y + y^2}{1 + y + y^2} = \dfrac{124}{7} \;\therefore$ Art. 106, $\dfrac{99 + 9y}{1 + y + y^2} = \dfrac{117}{7}$, or $\dfrac{11 + y}{1 + y + y^2} = \dfrac{13}{7}$ $\therefore 77 + 7y = 13 + 13y + 13y^2$, or $13y^2 + 6y = 64$; whence $y = 2$. Also $100x + 10xy + xy^2 + 594 = 100xy^2 + 10xy + x$; or $99x - 99xy^2 = -594$; or $x - xy^2 = -6$; or $x - 4x = -6$; or $-3x = -6 \;\therefore\; x = 2$, hence $100x + 10xy + xy^2 = 248$, the number required.

Exercise LXI.

1. (I) $A.S. = 7, 5, 3$; hence $d = -2 \;\therefore\; 13, 11, 9, 7, 5, 3, 1, -1, -3$ inverted, give $H.S.$ $\frac{1}{13}, \frac{1}{11}, \frac{1}{9}, \frac{1}{7}, \frac{1}{6}, \frac{1}{3}, 1, -1, -\frac{1}{3}$

(II) $A.S. = 18, 14, 10$; hence $d = -4 \;\therefore\; 30, 26, 22, 18, 14, 10, 6, 2, -2$ inverted, give $H.S.$ $\frac{1}{30}, \frac{1}{26}, \frac{1}{22}, \frac{1}{18}, \frac{1}{14}, \frac{1}{10}, \frac{1}{6}, \frac{1}{2}, -\frac{1}{2}$

(iii) $A.S. = 2, 4, 6$; hence $d = 2$ and $-4, -2, 0, 2, 4, 6, 8, 10, 12$ inverted, give $H.S.\ -\frac{1}{4}, -\frac{1}{2}, \infty, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}$

(iv) $A.S. = \frac{1}{14}, \frac{9}{14}, \frac{17}{14}$; hence $d = \frac{8}{14}$ and $-\frac{23}{14}, -\frac{15}{14}, -\frac{7}{14}, \frac{1}{14}, \frac{9}{14}, \frac{17}{14}, \frac{25}{14}, \frac{33}{14}$ and $\frac{41}{14}$ inverted, give $H.S.\ -\frac{14}{23}, -\frac{14}{15}, -2, 14, 1\frac{5}{9}, \frac{14}{17}, \frac{14}{25}, \frac{14}{33}$ and $\frac{14}{41}$

(v) $A.S. = \frac{11}{5}, \frac{4}{5}, -\frac{3}{5}$; hence $d = -\frac{7}{5}$ and $\frac{32}{5}, \frac{25}{5}, \frac{18}{5}, \frac{11}{5}, \frac{4}{5}, -\frac{3}{5}, -\frac{10}{5}$ and $-\frac{17}{5}$ inverted, give $H.S. = \frac{5}{32}, \frac{1}{5}, \frac{5}{18}, \frac{5}{11}, 1\frac{1}{4}, -1\frac{2}{3}, -\frac{1}{2}, -\frac{5}{17}, -\frac{5}{24}$

(vi) $A.S. = -2, 0, +2$; hence $d = 2$; then $-8, -6, -4, -2, 0, 2, 4, 6, 8$ inverted, give $H.S. = -\frac{1}{8}, -\frac{1}{6}, -\frac{1}{4}, -\frac{1}{2}, \infty, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}$

2. (i) Insert 3 $A.$ means, between $\frac{1}{2}$ and $\frac{1}{3}$. Here $d = \dfrac{\frac{1}{3} - \frac{1}{2}}{5 - 1} = \dfrac{-\frac{1}{6}}{4} = -\frac{1}{24}$; hence $A.$ series $= \frac{1}{2}, \frac{11}{24}, \frac{10}{24}, \frac{9}{24}, \frac{8}{24} = \frac{1}{2}, \frac{11}{24}, \frac{5}{12}, \frac{3}{8}, \frac{1}{3}$, and $\therefore H.S. = 2, \frac{24}{11}, \frac{12}{5}, \frac{8}{3}, \frac{3}{1} = 2, 2\frac{2}{11}, 2\frac{2}{5}, 2\frac{2}{3}, 3$

(ii) Insert 3 $A.$ means, between $\frac{1}{5}$ and $\frac{1}{7}$. Here $d = \dfrac{\frac{1}{7} - \frac{1}{5}}{5 - 1} = \dfrac{-\frac{2}{35}}{4} = -\frac{1}{70}$; hence $A.S. = \frac{14}{70}, \frac{13}{70}, \frac{12}{70}, \frac{11}{70}$ and $\frac{10}{70}$, and inverting these we have $H.S. = 5, 5\frac{5}{13}, 5\frac{5}{6}, 6\frac{4}{11}$ and 7

(iii) Insert 3 $A.$ means, between $\frac{1}{11}$ and $\frac{1}{3}$. Here $d = \dfrac{\frac{1}{3} - \frac{1}{11}}{5 - 1} = \dfrac{\frac{8}{33}}{4} = \frac{2}{33}$; hence $A.S. = \frac{3}{33}, \frac{5}{33}, \frac{7}{33}, \frac{9}{33}, \frac{11}{33}$, and inverting these we have $H.S. = 11, 6\frac{3}{5}, 4\frac{5}{7}, 3\frac{2}{3}, 3$

(iv) Insert 3 $A.$ means, between $\frac{4}{9}$ and $\frac{7}{22}$. Here $d = \dfrac{\frac{7}{22} - \frac{4}{9}}{5 - 1} = \dfrac{-\frac{25}{198}}{4} = -\frac{25}{792}$; hence $A.S. = \frac{352}{792}, \frac{327}{792}, \frac{302}{792}, \frac{277}{792}, \frac{252}{792}$ and inverting these, we have $H.S. = 2\frac{1}{4} + 2\frac{46}{109} + 2\frac{94}{151} + 2\frac{238}{277} + 3\frac{1}{7}$

(v) Insert 3 $A.$ means, between $\frac{1}{6}$ and $-\frac{5}{2}$. Here $d = \dfrac{-\frac{5}{2} - \frac{1}{6}}{5 - 1} = \dfrac{-\frac{8}{3}}{4} = -\frac{2}{3}$; hence $A.S. = \frac{1}{6}, -\frac{3}{6}, -\frac{7}{6}, -\frac{11}{6}, -\frac{15}{6}$, and inverting these, we have $H.S. = 6, -2, -\frac{6}{7}, -\frac{6}{11}, -\frac{2}{5}$

3. Corresponding $A.S. = \frac{2}{5}, 1, \frac{8}{5}$. Hence $d = \frac{3}{5}$; 5^{th} term of $A.S. = \frac{2}{5} + (5 - 1)\frac{3}{5} = \frac{2}{5} + \frac{12}{5} = \frac{14}{5}$; hence 5^{th} term of $H.S. = \frac{5}{14}$;

11^{th} term of $A.S. = \frac{2}{5} + (11-1)\frac{3}{5} = \frac{2}{5} + \frac{30}{5} = \frac{32}{5}$; hence 11^{th} term of $H.S. = \frac{5}{32}$; n^{th} term of $A.S. = \frac{2}{5} + (n-1)\frac{3}{5} = \frac{2}{5} + \frac{3n}{5} - \frac{3}{5} = \frac{3n-1}{5}$; hence n^{th} term of $H.S. = \frac{5}{3n-1}$

4. Of corresponding $A.S.\ \frac{3}{13}, \frac{2}{13}, \frac{1}{13}$ the 6^{th} term $= \frac{3}{13} + (6-1)\frac{1}{13} = \frac{3}{13} + \frac{5}{13} = \frac{8}{13}$; 10^{th} term $= \frac{3}{13} + (10-1)\frac{1}{13} = \frac{3}{13} + \frac{9}{13} = \frac{12}{13}$, and n^{th} term $= \frac{3}{13} + (n-1)\frac{1}{13} = \frac{3}{13} + \frac{n}{13} - \frac{1}{13} = \frac{2+n}{13}$ $\therefore$ required 6^{th}, 10^{th} and n^{th} terms of $H.S. = 1\frac{5}{8}$; $1\frac{1}{12}$ and $\frac{13}{n+2}$

5. Of the corresponding $A.S.$ 10, 12, 14, the 4^{th} term $= 10 + (4-1)2 = 16$, and the 8^{th} term $= 10 + (8-1)2 = 24$ $\therefore$ the 4^{th} and 8^{th} term of the $H.S. = \frac{1}{16}$ and $\frac{1}{24}$

6. Insert 2 $A.$ means, between $\frac{1}{4}$ and 1. Here $d = \frac{1-\frac{1}{4}}{4-1} = \frac{\frac{3}{4}}{3} = \frac{1}{4}$ hence $A.S. = \frac{1}{4} + \frac{1}{2} + \frac{3}{4} + \frac{4}{4}$, and inverting these, we have $H.S. = 4 + 2 + 1\frac{1}{3} + 1$; hence unknown terms are 2 and $1\frac{1}{3}$

7. Of the corresponding $A.S.\ \frac{1}{a}, \frac{1}{b}$, the 8^{th} term

$$= \frac{1}{a} + (8-1)\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{1}{a} + \frac{7}{b} - \frac{7}{a} = \frac{7}{b} - \frac{6}{a} = \frac{7a-6b}{ab};$$

hence 8^{th} term of $H.S. = \frac{ab}{7a-6b}$; n^{th} term $= \frac{1}{a} + (n-1)\left(\frac{1}{b} - \frac{1}{a}\right)$

$$= \frac{1}{a} + n\left(\frac{1}{b} - \frac{1}{a}\right) - \frac{1}{b} + \frac{1}{a} = \frac{2}{a} - \frac{1}{b} + \frac{n}{b} - \frac{n}{a} = \frac{2-n}{a} + \frac{n-1}{b}$$

$$= \frac{b(2-n) + a(n-1)}{ab} \quad \therefore\ n^{\text{th}} \text{ term of } H.S. = \frac{ab}{b(2-n) + a(n-1)}$$

8. $$H.M. = \frac{2ab}{a+b} = \frac{\frac{2}{m^2-n^2}}{\frac{1}{m+n} + \frac{1}{m-n}} = \frac{\frac{2}{m^2-n^2}}{\frac{2m}{m^2-n^2}} = \frac{1}{m}$$

9. $A.M. = \frac{1}{2}(a+b) = \frac{1}{2}(4+9) = \frac{13}{2} = 6\frac{1}{2}$; $G.M. = \sqrt{ab} = \sqrt{4 \times 9} = \sqrt{36} = 6$; $H.M. = \frac{2ab}{a+b} = \frac{2 \times 4 \times 9}{4+9} = \frac{72}{13} = 5\frac{7}{13}$

10. $A.M. = \frac{1}{2}(6 + 4\frac{1}{6}) = \frac{1}{2}$ of $10\frac{1}{6} = 5\frac{1}{12}$; $G.M. = \sqrt{6 \times 4\frac{1}{6}} = \sqrt{25}$ $= 5$; $H.M. = \dfrac{2 \times 6 \times 4\frac{1}{6}}{6 + 4\frac{1}{6}} = \dfrac{50}{10\frac{1}{6}} = \frac{300}{61} = 4\frac{56}{61}$

11. $a : c :: a - b : b - c \therefore ab - ac = ac - bc \therefore 2ac = ab + bc = b(a + c)$ $\therefore b = \dfrac{2ac}{a + c} \therefore 2b^2 = 2\left(\dfrac{2ac}{a + c}\right)^2 \therefore a^2 + c^2 - 2b^2 = a^2 + c^2 - 2\left(\dfrac{2ac}{a + c}\right)^2$ $= a^2 - 2ac + c^2 + 2ac - 2\left(\dfrac{2ac}{a + c}\right)^2 = (a - c)^2 + 2ac\left\{1 - \dfrac{4ac}{(a + c)^2}\right\}$ $= (a - c)^2 + 2ac\left\{\dfrac{(a - c)^2}{(a + c)^2}\right\}$ = a positive quantity if a and c have like signs $\therefore a^2 + c^2 > 2b^2$

12. $b = \frac{1}{2}(a + c)$, and $mb = \sqrt{ac}$; substituting the value of b, we have $\dfrac{m}{2}(a + c) = \sqrt{ac} \therefore \dfrac{m^2}{4}(a + c)^2 = ac \therefore m^2(a + c)^2 = 4ac$, and dividing each by $a + c$ we get $m^2(a + c) = \dfrac{4ac}{a + c}$, but $a + c = 2b$ $\therefore 2bm^2 = \dfrac{4ac}{a + c}$, or $bm^2 = \dfrac{2ac}{a + c}$; hence Art. 261, bm^2 is the *H.M.* between a and $c \therefore a$, bm^2 and c are in H. Prog.

13. Let a, b and c be any three quantities in H. Prog., and let x be the quantity which, when subtracted from each, leaves remainders in *G.P.*; then $(a - x)(c - x) = (b - x)^2$, that is $ac - cx - ax + x^2 = b^2 - 2bx + x^2 \therefore 2bx - cx - ax = b^2 - ac$ $\therefore x = \dfrac{b^2 - ac}{2b - c - a}$; but since a, b and c are in *H.P.*, $a : c :: a - b : b - c \therefore ab - ac = ac - bc$; $ab = 2ac - bc$ $\therefore c = \dfrac{ab}{2a - b}$. Substitute this for c in the above value of x,

and we have $x = \dfrac{b^2 - \dfrac{a^2b}{2a - b}}{2b - a - \dfrac{ab}{2a - b}} = \dfrac{\dfrac{2ab^2 - b^3 - a^2b}{2a - b}}{\dfrac{4ab - 2a^2 - 2b^2 + ab - ab}{2a - b}}$

$$x = \frac{2ab^2 - b^3 - a^2b}{4ab - 2b^2 - 2a^2} = \frac{b(2ab - b^2 - a^2)}{2(2ab - b^2 - a^2)} = \frac{b}{2} = \tfrac{1}{2} \text{ of middle term.}$$

14. $A.M. = \frac{1}{2}(a + b)$, and $G.M. = \sqrt{ab}$ $\therefore \frac{1}{2}(a + b) + \sqrt{ab} = 16$ (I) and $\frac{1}{2}(a + b) - \sqrt{ab} = 4$ (II). From (I) $a + 2\sqrt{ab} + b = 32$ $\therefore \sqrt{a} + \sqrt{b} = \pm 4\sqrt{2}$. From (II) $a - 2\sqrt{ab} + b = 8$ $\therefore \sqrt{a} - \sqrt{b} = \pm 2\sqrt{2}$ $\therefore 2\sqrt{a} = \pm 6\sqrt{2}$, whence $a = 18$; and $2\sqrt{b} = \pm 2\sqrt{2}$, whence $b = 2$

15. $A.M. = \frac{1}{2}(a + b)$, and $H.M. = \dfrac{2ab}{a + b}$ $\therefore \frac{1}{2}(2 + b) = \frac{16}{7} \times \dfrac{4b}{2 + b}$ $= \dfrac{64b}{14 + 7b}$ $\therefore 28 + 14b + 14b + 7b^2 = 128b$ $\therefore 7b^2 - 100b = -28$, or $b^2 - \frac{100}{7}b = -4$; $b^2 - \frac{100}{7}b + \frac{2500}{49} = \dfrac{2500 - 196}{49} = \frac{2304}{49}$ $\therefore b - \frac{50}{7} = \pm \frac{48}{7}$ $\therefore b = 14$ or $\frac{2}{7}$

16. $a + b = 30$, and $\dfrac{2ab}{a + b} = 13\frac{1}{3}$ $\therefore \dfrac{2ab}{30} = 13\frac{1}{3}$ $\therefore 2ab = 400$; $a^2 + 2ab + b^2 = 900$, and $4ab = 800$ $\therefore a^2 - 2ab + b^2 = 100$, or $a - b = \pm 10$; $a + b = 30$, and $a - b = \pm 10$ $\therefore 2a = 40$, or 20; $a = 20$ or 10; $2b = 20$ or 40 $\therefore b = 10$ or 20 $\therefore$ the numbers are 20 and 10

17. $a - b = 16\frac{1}{4}$, and $\sqrt{ab} = 9$, since the $G.M.$ between the $A.$ and $H.M.$ of a and b = $G.M.$ between a and b, (see Art. 261) Then $a^2 - 2ab + b^2 = \frac{4225}{16}$, and $ab = 81$ $\therefore 4ab = 324$; $a^2 + 2ab + b^2 = \frac{4225}{16} + 324 = \frac{9409}{16}$ $\therefore a + b = \frac{97}{4}$. Hence $a - b = \frac{65}{4}$, and $a + b = \frac{97}{4}$ $\therefore 2a = \frac{81}{2}$ $\therefore a = \frac{81}{4} = 20\frac{1}{4}$; $2b = \frac{32}{4} = 8$ $\therefore b = 4$

EXERCISE LXII.

1. $V_6 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$

2. (I) $V_4 = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$; (II) $V_6 = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 20160$; $V_8 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 = 40320$

3. We are to find the permutations of 13 letters of which 5 are a's, 4 are b's, and 3 are c's

Then $N = \dfrac{\underline{|n}}{\underline{|p}\;\underline{|q}\;\underline{|r}} = \dfrac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \times 1 \cdot 2 \cdot 3 \cdot 4 \times 1 \cdot 2 \cdot 3} = 360360$

4. $V_{12} = 1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10\cdot11\cdot12$ = whole number of changes $\frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10\cdot11\cdot12}{16\cdot60\cdot10} = 49896$ = number of days required = 136 years 222 days.

5. $V_5 = n(n-1)(n-2)(n-3)(n-4)$, and $V_3 = n(n-1)(n-2)$ Then $n(n-1)(n-2)(n-3)(n-4) = 6 \times n(n-1)(n-2)$ $\therefore (n-3)(n-4) = 6$; that is $n^2 - 7n = -6$, whence $n = 6$

6. $V_{10} = 1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10$ = whole number of days $\therefore 1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot8\cdot9\cdot10 = 518400$ = number of weeks he had to board them, and since board is worth \$5 per week for one person, it is worth \$50 per week for 10. Hence total value of board = \$50 × 518400 = \$25920000; and \$25920000 − \$5000 = \$25915000 = loss when the \$5000 is not paid till the expiration of the term of the board. And amount of \$5000 at 6 per cent. for $\frac{3628800}{365\frac{1}{4}}$ years, i. e. for 9935·112 years = $5000(1 + rt)$ = $5000(1 + 596\cdot112) = 5000 \times 597\cdot112$ = \$2985533·60. Hence his loss when the \$5000 is paid at once, and put out at interest until the expiration of the term = \$25920000 − \$2985533·60 = \$22934466·40

7. $V_n = 15\cdot14\cdot13\cdots\cdots(15-n+2)(15-n+1)$ and $V_{n-1} = 15\cdot14\cdot13\cdots\cdots\{15-(n-1)+1\}$; then $15\cdot14\cdot13\cdots\cdot(15-n+2)(15-n+1) = 15\cdot14\cdot13\cdots\cdot(15-n+2)\times10$ $\therefore$ cancelling same factors of both sides, we have $15 - n + 1 = 10$ $\therefore n = 6$

8. (I) Permutations of 14 letters whereof 2 are o's, 3 are n's, and two are t's $= \frac{\lfloor n}{\lfloor p \; \lfloor q \; \lfloor r} = \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10\cdot11\cdot12\cdot13\cdot14}{1\cdot2 \times 1\cdot2\cdot3 \times 1\cdot2}$ = 3632428800

(II) Permutations of 12 letters whereof 5 are i's

$$= \frac{\lfloor n}{\lfloor p} = \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10\cdot11\cdot12}{1\cdot2\cdot3\cdot4\cdot5} = 3991680$$

(III) Permutations of 8 letters whereof 4 are o's

$$= \frac{\underline{|n}}{\underline{|p}} = \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}{1\cdot2\cdot3\cdot4} = 1680$$

(IV) Permutations of 13 letters whereof 3 are o's and 3 are m's

$$= \frac{\underline{|n}}{\underline{|p}\ \underline{|q}} = \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10\cdot11\cdot12\cdot13}{1\cdot2\cdot3 \times 1\cdot2\cdot3} = 172972800$$

9. (I) Permutations of 7 letters of which 2 are a's

$$= \frac{\underline{|n}}{\underline{|p}} = \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}{1\cdot2} = 2520$$

(II) Permutations of 13 letters whereof 2 are o's, 2 are n's, and 2 are t's $= \dfrac{\underline{|n}}{\underline{|p}\ \underline{|q}\ \underline{|r}} = \dfrac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10\cdot11\cdot12\cdot13}{1\cdot2 \times 1\cdot2 \times 1\cdot2} = 778377600$

(III) Permutations of 7 letters whereof 2 are t's and 3 are o's

$$= \frac{\underline{|n}}{\underline{|p}\ \underline{|q}} = \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}{1\cdot2 \times 1\cdot2\cdot3} = 420$$

10. $\dfrac{5n}{2}\left(\dfrac{5n}{2} - 1\right)\left(\dfrac{5n}{2} - 2\right) : \dfrac{2n}{3}\left(\dfrac{2n}{3} - 1\right)\left(\dfrac{2n}{3} - 2\right) :: 145 : 2$

$$\therefore\ 5n\left(\frac{5n-2}{2}\right)\left(\frac{5n-4}{2}\right) = \frac{290n}{3}\left(\frac{2n-3}{3}\right)\left(\frac{2n-6}{3}\right);$$

or $\frac{5}{4}(5n-2)(5n-4) = \frac{290}{27}(2n-3)(2n-6)$; or $135(25n^2 - 30n + 8) = 2320(2n^2 - 9n + 9)$; or $253n^2 - 3366n = -3960$, whence $n = 12$

Exercise LXIII.

1. (I) $C_3 = \dfrac{10\cdot9\cdot8}{1\cdot2\cdot3} = 120$; (II) $C_5 = \dfrac{10\cdot9\cdot8\cdot7\cdot6}{1\cdot2\cdot3\cdot4\cdot5} = 252$;

(III) $C_8 = C_2 = \dfrac{10\cdot9}{1\cdot2} = 45$

2. (I) $C_5 = \dfrac{15\cdot14\cdot13\cdot12\cdot11}{1\cdot2\cdot3\cdot4\cdot5} = 3003$;

(II) $C_7 = \dfrac{15\cdot14\cdot13\cdot12\cdot11\cdot10\cdot9}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7} = 6435$;

(III) $C_{12} = C_3 = \dfrac{15\cdot14\cdot13}{1\cdot2\cdot3} = 455$

3. $C_5 = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 792$

4. Whole number of combinations of $2n$ things, 1, 2, 3, 4, &c., ... $2n$ together $= 2^{2n} - 1$; similarly the whole number of combinations of n things, 1, 2, 3, 4, &c., .. n together $= 2^n - 1$. Then $2^{2n} - 1 = (2^n - 1) \times 513 \therefore \frac{2^{2n} - 1}{2^n - 1} = 513$; or since $\frac{2^{2n} - 1}{2^n - 1} = 2^n + 1$, we have $2^n + 1 = 513 \therefore 2^n = 512$, and $\therefore$ by inspection $n = 8$

5. (I) $C_5 = \frac{36 \cdot 35 \cdot 34 \cdot 33 \cdot 32}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 439824$ = No. of different selections

(II) Taking away one man from the 36 there remain 35, and these combined together, 4 and 4 give $\frac{35 \cdot 34 \cdot 32 \cdot 31}{1 \cdot 2 \cdot 3 \cdot 4} = 52360$ combinations to each of which the reserved man must be attached.

6. Number of combinations of 21 consonants, 4 together $= \frac{21 \cdot 20 \cdot 19 \cdot 18}{1 \cdot 2 \cdot 3 \cdot 4} = 5985$; also number of combinations of 5 vowels, 3 together $= \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = 10$. Hence there can be formed $5985 \times 10 = 59850$ *different* sets of seven letters, each set containing four consonants and three vowels. But each of these 59850 sets can be permutated, $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 5040$ ways, each forming a different word $\therefore$ the required number of words $= 59850 \times 5040 = 301644000$.

7. The different arrangements of 9 of the persons while the tenth remains fixed $= 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362880$ = whole number of different arrangements of the ten persons, so that no one has the same neighbours in any two cases. But one half of these arrangements will be similar to the other half if the position of neighbours on the right and left hand sides be not regarded as making a difference. So that if A is said to have the *same* neighbours in the arrangement BAC that he has in the arrangement CAB, then the correct answer will be $\frac{1}{2}$ of $362880 = 181440$

8. $n(n-1)(n-2) : \frac{n(n-1)(n-2)(n-3)}{1\cdot2\cdot3\cdot4} :: 6 : 1 \therefore$ cancelling, $\frac{n-3}{4} = 1$, whence $n = 7$

9. $n(n-1)(n-2) \ldots (n-p+1) = 10n(n-1)(n-2) \ldots (n-p+2)$; or dividing each by $n(n-1)(n-2) \ldots (n-p+2)$, we get $n-p+1 = 10 \therefore n-p = 9$ (I). Again

$$\frac{n(n-1)(n-2) \ldots (n-p+1)}{\lfloor p} : \frac{n(n-1)(n-2) \ldots (n-p+2)}{\lfloor p-1} :: 5 : 3$$

or multiplying each side by $\lfloor p-1$ we have

$$\frac{n(n-1)(n-2) \ldots (n-p+1)}{p} = \tfrac{5}{3}n(n-1)(n-2) \ldots (n-p+2),$$

and dividing each side by $n(n-1)(n-2) \ldots (n-p+2)$, we have $\frac{n-p+1}{p} = \frac{5}{3}$; or $3n-3p+3 = 5p \therefore 3n-8p = -3$ (II). Now multiplying (I) by 3, and subtracting from (II), we have $5p = 30 \therefore p = 6$, and similarly $n = 15$

10. $1\cdot2\cdot3 \cdots (n-1) = \lfloor n-1$, or $\frac{1}{2}\lfloor n-1$ according as BAC and CAB are regarded as different or the same arrangement.

11. Number of 5 flag signals with 10 flags $= \frac{10\cdot9\cdot8\cdot7\cdot6}{1\cdot2\cdot3\cdot4\cdot5} = 252$; number of signals with four flags out of 10 $= \frac{10\cdot9\cdot8\cdot7}{1\cdot2\cdot3\cdot4} = 210$; number with 3 flags $= \frac{10\cdot9\cdot8}{1\cdot2\cdot3} = 120$; number with 2 flags $= \frac{10\cdot9}{1\cdot2} = 45$ and number with one flag $= 10$. Therefore whole number of signals $= 10 + 45 + 120 + 210 + 252 = 637$

12. There are in all nine coins and they may be combined, any number together, to make a sum; then the combinations of 9 things 1, 2, 3, ... 9 together $= 2^n - 1 = 2^9 - 1 = 512 - 1 = 511$

Exercise LXIV.

1. $(1+x)^{-3} = 1 - \frac{3}{1}x + \frac{3\cdot4}{1\cdot2}x^2 - \frac{3\cdot4\cdot5}{1\cdot2\cdot3}x^3 + \frac{3\cdot4\cdot5\cdot6}{1\cdot2\cdot3\cdot4}x^4 - \&c.$

$= 1 - 3x + 6x^2 - 10x^3 + 15x^4 - \&c.$

2. $(1+x)^{-2} = 1 - \frac{2}{1}x + \frac{2\cdot3}{1\cdot2}x^2 - \frac{2\cdot3\cdot4}{1\cdot2\cdot3}x^3 + \frac{2\cdot3\cdot4\cdot5}{1\cdot2\cdot3\cdot4}x^4 - \&c.$

$= 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \&c.$

3. $(1-2x)^{-1} = 1 + \frac{1}{1}(2x) + \frac{1\cdot2}{1\cdot2}(2x)^2 + \frac{1\cdot2\cdot3}{1\cdot2\cdot3}(2x)^3 + \frac{1\cdot2\cdot3\cdot4}{1\cdot2\cdot3\cdot4}(2x)^4 + \&c.$

$= 1 + 2x + 4x^2 + 8x^3 + 16x^4 + \&c.$

4. $(1-\frac{1}{2}x)^{-5} = 1 + \frac{5}{1}(\frac{1}{2}x) + \frac{5\cdot6}{1\cdot2}(\frac{1}{2}x)^2 + \frac{5\cdot6\cdot7}{1\cdot2\cdot3}(\frac{1}{2}x)^3 + \frac{5\cdot6\cdot7\cdot8}{1\cdot2\cdot3\cdot4}(\frac{1}{2}x)^4$

$+ \&c. = 1 + \frac{5}{2}x + \frac{15}{4}x^2 + \frac{35}{8}x^3 + \frac{35}{8}x^4 + \&c.$

5. $(1+3x)^{-2} = 1 - \frac{2}{1}(3x) + \frac{2\cdot3}{1\cdot2}(3x)^2 - \frac{2\cdot3\cdot4}{1\cdot2\cdot3}(3x)^3 + \frac{2\cdot3\cdot4\cdot5}{1\cdot2\cdot3\cdot4}(3x)^4$

$- \&c. = 1 - 6x + 27x^2 - 108x^3 + 405x^4 - \&c.$

6. $(1-2x)^{-5} = 1 + \frac{5}{1}(2x) + \frac{5\cdot6}{1\cdot2}(2x)^2 + \frac{5\cdot6\cdot7}{1\cdot2\cdot3}(2x)^3 + \frac{5\cdot6\cdot7\cdot8}{1\cdot2\cdot3\cdot4}(2x)^4$

$+ \&c. = 1 + 10x + 60x^2 + 280x^3 + 1120x^4 + \&c.$

7. $(1-x)^{-4} = 1 + \frac{4}{1}x + \frac{4\cdot5}{1\cdot2}x^2 + \frac{4\cdot5\cdot6}{1\cdot2\cdot3}x^3 + \frac{4\cdot5\cdot6\cdot7}{1\cdot2\cdot3\cdot4}x^4 + \&c.$

$= 1 + 4x + 10x^2 + 20x^3 + 35x^4 + \&c.$

8. $(1-4x)^{\frac{1}{2}} = 1 - \frac{1}{2}(4x) + \frac{1\cdot(-1)}{1\cdot2\cdot4}(4x)^2 - \frac{1\cdot(-1)(-3)}{1\cdot2\cdot3\cdot8}(4x)^3$

$+ \frac{1\cdot(-1)(-3)(-5)}{1\cdot2\cdot3\cdot4\cdot16}(4x)^4 - \&c. = 1 - 2x - 2x^2 - 4x^3 - 10x^4 - \&c.$

9. $(1+x)^{-\frac{2}{3}} = 1 - \frac{2}{3}x + \frac{2\cdot5}{1\cdot2\cdot9}x^2 - \frac{2\cdot5\cdot8}{1\cdot2\cdot3\cdot27}x^3 + \frac{2\cdot5\cdot8\cdot11}{1\cdot2\cdot3\cdot4\cdot81}x^4 - \&c.$

$= 1 - \frac{2}{3}x + \frac{5}{9}x^2 - \frac{40}{81}x^3 + \frac{110}{243}x^4 - \&c.$

10. $(1-\frac{3}{4}x)^{\frac{4}{5}} = 1 - \frac{4}{5}(\frac{3}{4}x) - \frac{4(-1)}{1\cdot2\cdot25}(\frac{3}{4}x)^2 - \frac{4(-1)(-6)}{1\cdot2\cdot3\cdot125}(\frac{3}{4}x)^3$

$+ \frac{4(-1)(-6)(-11)}{1\cdot2\cdot3\cdot4\cdot625}(\frac{3}{4}x)^4 - \&c.$

$= 1 - \frac{3}{5}x - \frac{9}{200}x^2 - \frac{27}{2000}x^3 - \frac{891}{160000}x^4 - \&c.$

11. $(1+\frac{2}{3}x)^{\frac{1}{3}} = 1 + \frac{1}{3}(\frac{2}{3}x) + \frac{1(-2)}{1\cdot2\cdot9}(\frac{2}{3}x)^2 + \frac{1(-2)(-5)}{1\cdot2\cdot3\cdot27}(\frac{2}{3}x)^3$
$+ \frac{1(-2)(-5)(-8)}{1\cdot2\cdot3\cdot4\cdot81}(\frac{2}{3}x)^4 + \&c.$
$= 1 + \frac{2}{9}x - \frac{4}{81}x^2 + \frac{40}{2187}x^3 - \frac{160}{19683}x^4 + \&c.$

12. $(1-x)^{-\frac{4}{5}} = 1 + \frac{4}{5}x + \frac{4\cdot9}{1\cdot2\cdot25}x^2 + \frac{4\cdot9\cdot14}{1\cdot2\cdot3\cdot125}x^3 + \frac{4\cdot9\cdot14\cdot19}{1\cdot2\cdot3\cdot4\cdot625}x^4$
$+ \&c. = 1 + \frac{4}{5}x + \frac{18}{25}x^2 + \frac{84}{125}x^3 + \frac{399}{625}x^4 + \&c.$

13. $(a-x^2)^{-3} = \{a(1-a^{-1}x^2)\}^{-3} = a^{-3}(1-a^{-1}x^2)^{-3}$
$= a^{-3}\{1 + \frac{3}{1}(a^{-1}x^2) + \frac{3\cdot4}{1\cdot2}(a^{-1}x^2)^2 + \frac{3\cdot4\cdot5}{1\cdot2\cdot3}(a^{-1}x^2)^3 + \frac{3\cdot4\cdot5\cdot6}{1\cdot2\cdot3\cdot4}(a^{-1}x^2)^4 + \&c.\}$
$= a^{-3}\{1 + 3a^{-1}x^2 + 6a^{-2}x^4 + 10a^{-3}x^6 + 15a^{-4}x^8 + \&c.\}$
$= a^{-3} + 3a^{-4}x^2 + 6a^{-5}x^4 + 10a^{-6}x^6 + 15a^{-7}x^8 + \&c.$

14. $(a^2+x^3)^{-1} = \{a^2(1+a^{-2}x^3)\}^{-1} = a^{-2}(1-a^{-2}x^3)^{-1}$
$= a^{-2}\{1 - \frac{1}{1}(a^{-2}x^3) + \frac{1\cdot2}{1\cdot2}(a^{-2}x^3)^2 - \frac{1\cdot2\cdot3}{1\cdot2\cdot3}(a^{-2}x^3)^3 + \frac{1\cdot2\cdot3\cdot4}{1\cdot2\cdot3\cdot4}(a^{-2}x^3)^4 + \&c.\}$
$= a^{-2}(1 - a^{-2}x^3 + a^{-4}x^6 - a^{-6}x^9 + a^{-8}x^{12} - \&c.)$
$= a^{-2} - a^{-4}x^3 + a^{-6}x^6 - a^{-8}x^9 + a^{-10}x^{12} - \&c.$

15. $(a^{\frac{1}{2}} - x^{\frac{1}{3}})^{-2} = \{a^{\frac{1}{2}}(1 - a^{-\frac{1}{2}}x^{\frac{1}{3}})\}^{-2} = a^{-1}(1-a^{-\frac{1}{2}}x^{\frac{1}{3}})^{-2}$
$= a^{-1}\{1 + \frac{2}{1}(a^{-\frac{1}{2}}x^{\frac{1}{3}}) + \frac{2\cdot3}{1\cdot2}(a^{-\frac{1}{2}}x^{\frac{1}{3}})^2 + \frac{2\cdot3\cdot4}{1\cdot2\cdot3}(a^{-\frac{1}{2}}x^{\frac{1}{3}})^3$
$+ \frac{2\cdot3\cdot4\cdot5}{1\cdot2\cdot3\cdot4}(a^{-\frac{1}{2}}x^{\frac{1}{3}})^4 + \&c.\}$
$= a^{-1}\{1 + 2a^{-\frac{1}{2}}x^{\frac{1}{3}} + 3a^{-1}x^{\frac{2}{3}} + 4a^{-\frac{3}{2}}x + 5a^{-2}x^{\frac{4}{3}} + \&c.\}$
$= a^{-1} + 2a^{-\frac{3}{2}}x^{\frac{1}{3}} + 3a^{-2}x^{\frac{2}{3}} + 4a^{-\frac{5}{2}}x + 5a^{-3}x^{\frac{4}{3}} + \&c.$

16. $(a^4 - x^3)^{\frac{2}{3}} = \{a^4(1-a^{-4}x^3)\}^{\frac{2}{3}} = a^{\frac{8}{3}}(1-a^{-4}x^3)^{\frac{2}{3}}$
$= a^{\frac{8}{3}}\{1 - \frac{2}{3}(a^{-4}x^3) + \frac{2(-1)}{1\cdot2\cdot9}(a^{-4}x^3)^2 - \frac{2(-1)(-4)}{1\cdot2\cdot3\cdot27}(a^{-4}x^3)^3$
$+ \frac{2(-1)(-4)(-7)}{1\cdot2\cdot3\cdot4\cdot81}(a^{-4}x^3)^4 - \&c.\}$
$= a^{\frac{8}{3}}\{1 - \frac{2}{3}a^{-4}x^3 + \frac{1}{9}a^{-8}x^6 - \frac{4}{81}a^{-12}x^9 - \frac{7}{243}a^{-16}x^{12} - \&c.\}$
$= a^{\frac{8}{3}} - \frac{2}{3}a^{-\frac{4}{3}}x^3 - \frac{1}{9}a^{-\frac{16}{3}}x^6 - \frac{4}{81}a^{-\frac{28}{3}}x^9 - \frac{7}{243}a^{-\frac{40}{3}}x^{12} - \&c.$

17. $(a^3 + x^{-2})^{-4} = \{a^3(1 + a^{-3}x^{-2})\}^{-4} = a^{-12}(1 + a^{-3}x^{-2})^{-4}$

$$= a^{-12}\{1 - \frac{4}{1}(a^{-3}x^{-2}) + \frac{4\cdot5}{1\cdot2}(a^{-3}x^{-2})^2 - \frac{4\cdot5\cdot6}{1\cdot2\cdot3}(a^{-3}x^{-2})^3 + \frac{4\cdot5\cdot6\cdot7}{1\cdot2\cdot3\cdot4}(a^{-3}x^{-2})^4 - \&c.\}$$

$$= a^{-12}(1 - 4a^{-3}x^{-2} + 10a^{-6}x^{-4} - 20a^{-9}x^{-6} + 35a^{-12}x^{-8} - \&c.)$$

$$= a^{-12} - 4a^{-15}x^{-2} + 10a^{-18}x^{-4} - 20a^{-21}x^{-6} + 35a^{-24}x^{-8} - \&c$$

18. $\left(a^{\frac{1}{5}} - x^{-\frac{1}{5}}\right)^{-\frac{1}{3}} = \left\{a^{\frac{1}{5}}\left(1 - a^{-\frac{1}{5}}x^{-\frac{1}{5}}\right)\right\}^{-\frac{1}{3}} = a^{-\frac{1}{15}}\left\{1 - a^{-\frac{1}{5}}x^{-\frac{1}{5}}\right\}^{-\frac{1}{3}}$

$$= a^{-\frac{1}{15}}\{1 + \frac{1}{3}(ax)^{-\frac{1}{5}} + \frac{1\cdot4}{1\cdot2\cdot9}(ax)^{-\frac{2}{5}} + \frac{1\cdot4\cdot7}{1\cdot2\cdot3\cdot27}(ax)^{-\frac{3}{5}} + \frac{1\cdot4\cdot7\cdot10}{1\cdot2\cdot3\cdot4\cdot81}(ax)^{-\frac{4}{5}} + \&c.\}$$

$$= a^{-\frac{1}{15}}\{1 + \tfrac{1}{3}(ax)^{-\frac{1}{5}} + \tfrac{2}{9}(ax)^{-\frac{2}{5}} + \tfrac{14}{81}(ax)^{-\frac{3}{5}} + \tfrac{35}{243}(ax)^{-\frac{4}{5}} + \&c.\}$$

$$= a^{-\frac{1}{15}} + \tfrac{1}{3}a^{-\frac{4}{15}}x^{-\frac{1}{5}} + \tfrac{2}{9}a^{-\frac{7}{15}}x^{-\frac{2}{5}} + \tfrac{14}{81}a^{-\frac{2}{3}}x^{-\frac{3}{5}} + \tfrac{35}{243}a^{-\frac{13}{15}}x^{-\frac{4}{5}} + \&c.$$

19. $\left(a^2m - x^{\frac{1}{2}}\right)^{-\frac{2}{3}} = \left\{a^2m(1 - a^{-2}m^{-1}x^{\frac{1}{2}})\right\}^{-\frac{2}{3}}$

$$= a^{-\frac{4}{3}}m^{-\frac{2}{3}}\left(1 - a^{-2}m^{-1}x^{\frac{1}{2}}\right)^{-\frac{2}{3}}$$

$$= a^{-\frac{4}{3}}m^{-\frac{2}{3}}\left\{1 + \frac{2}{3}\left(\frac{x^{\frac{1}{2}}}{a^2m}\right) + \frac{2\cdot5}{1\cdot2\cdot9}\left(\frac{x^{\frac{1}{2}}}{a^2m}\right)^2 + \frac{2\cdot5\cdot8}{1\cdot2\cdot3\cdot27}\left(\frac{x^{\frac{1}{2}}}{a^2m}\right)^3 + \frac{2\cdot5\cdot8\cdot11}{1\cdot2\cdot3\cdot4\cdot81}\left(\frac{x^{\frac{1}{2}}}{a^2m}\right)^4 + \&c.\right\}$$

$$= a^{-\frac{4}{3}}m^{-\frac{2}{3}}\{1 + \frac{2}{3}\left(\frac{x^{\frac{1}{2}}}{a^2m}\right) + \frac{5}{9}\left(\frac{x}{a^4m^2}\right) + \frac{40}{81}\left(\frac{x^{\frac{3}{2}}}{a^6m^3}\right) + \frac{110}{243}\left(\frac{x^2}{a^8m^4}\right) + \&c.\}$$

$$= a^{-\frac{4}{3}}m^{-\frac{2}{3}} + \tfrac{2}{3}a^{-\frac{10}{3}}m^{-\frac{5}{3}}x^{\frac{1}{2}} + \tfrac{5}{9}a^{-\frac{16}{3}}m^{-\frac{8}{3}}x + \tfrac{40}{81}a^{-\frac{22}{3}}m^{-\frac{11}{3}}x^{\frac{3}{2}} + \tfrac{110}{243}a^{-\frac{28}{3}}m^{-\frac{14}{3}}x^2 + \&c.$$

20. $(a + x^{-3})^{\frac{2}{5}} = \{a(1 + a^{-1}x^{-3})\}^{\frac{2}{5}} = a^{\frac{2}{5}}(1 + a^{-1}x^{-3})^{\frac{2}{5}}$

$$= a^{\frac{2}{5}}\{1 + \frac{2}{5}\left(\frac{1}{ax^3}\right) + \frac{2(-3)}{1\cdot2\cdot25}\left(\frac{1}{ax^3}\right)^2 + \frac{2(-3)(-8)}{1\cdot2\cdot3\cdot125}\left(\frac{1}{ax^3}\right)^3 + \frac{2(-3)(-8)(-13)}{1\cdot2\cdot3\cdot4\cdot625}\left(\frac{1}{ax^3}\right)^4 + \&c.\}$$

$$= a^{\frac{2}{5}}\left\{1 + \frac{2}{5}\left(\frac{1}{ax^3}\right) + \frac{3}{25}\left(\frac{1}{a^2x^6}\right) + \frac{8}{125}\left(\frac{1}{a^3x^9}\right) - \frac{26}{625}\left(\frac{1}{a^4x^{12}}\right) + \&c\right\}$$

$$= a^{\frac{2}{5}} + \tfrac{2}{5}a^{-\frac{3}{5}}x^{-3} - \tfrac{3}{25}a^{-\frac{8}{5}}x^{-6} + \tfrac{8}{125}a^{-\frac{13}{5}}x^{-9} - \tfrac{26}{625}a^{-\frac{18}{5}}x^{-12}$$

21. $(a - bx)^{-\frac{1}{2}} = \{a(1 - a^{-1}bx)\}^{-\frac{1}{2}} = a^{-\frac{1}{2}}(1 - a^{-1}bx)^{-\frac{1}{2}}$

$$= a^{-\frac{1}{2}}\left\{1 + \frac{1}{2}\left(\frac{bx}{a}\right) + \frac{1\cdot3}{1\cdot2\cdot4}\left(\frac{bx}{a}\right)^2 + \frac{1\cdot3\cdot5}{1\cdot2\cdot3\cdot8}\left(\frac{bx}{a}\right)^3 + \frac{1\cdot3\cdot5\cdot7}{1\cdot2\cdot3\cdot4\cdot16}\left(\frac{bx}{a}\right)^4 + \&c.\right\}$$

$$= a^{-\frac{1}{2}}\left\{1 + \frac{1}{2}\left(\frac{bx}{a}\right) + \frac{3}{8}\left(\frac{b^2x^2}{a^2}\right) + \frac{5}{16}\left(\frac{b^3x^3}{a^3}\right) + \frac{35}{128}\left(\frac{b^4x^4}{a^4}\right) + \&c.\right\}$$

$$= a^{-\frac{1}{2}} + \tfrac{1}{2}a^{-\frac{3}{2}}bx + \tfrac{3}{8}a^{-\frac{5}{2}}b^2x^2 + \tfrac{5}{16}a^{-\frac{7}{2}}b^3x^3 + \tfrac{35}{128}a^{-\frac{9}{2}}b^4x^4 \ \&c.$$

Exercise LXV.

1. Gen. term of $(1 - x)^{-3} = \dfrac{n(n+1)(n+2)\cdots\cdots(n+r-1)}{\lfloor r}x^r$

$$= \frac{3\cdot4\cdot5\cdots\cdots(2+r)}{\lfloor r}x^r$$

(II) Since general term = the $(r+1)^{\text{th}}$ term = 6^{th} term $\therefore r = 5$

Hence 6^{th} term $= \dfrac{3\cdot4\cdot5\cdot6\cdot7}{1\cdot2\cdot3\cdot4\cdot5}x^5 = 21x^5$

2. (I) Gen. term of $(1+x)^{-4} = (-1)^r \times \dfrac{n(n+1)(n+2)\cdots(n+r-1)}{\lfloor r}x^r$

$$= (-1)^r \times \frac{4\cdot5\cdot6\cdots\cdots(3+r)}{\lfloor r}x^r$$

(II) Since general term = 6^{th} term = $(1+r)^{\text{th}}$ term $\therefore r = 5$

Hence 6^{th} term $= (-1)^5 \times \dfrac{4\cdot5\cdot6\cdot7\cdot8}{1\cdot2\cdot3\cdot4\cdot5}x^5 = -56x^5$

3. (I) General term of $(1 - x)^{-\frac{2}{3}}$

$$= (-1)^r \times \frac{p(p+q)(p+2q)\cdots\cdots\cdots\{p+(r-1)q\}}{\lfloor r \times q^r}x^r$$

$$= (-1)^r \times \frac{2\cdot5\cdot8\cdots\cdots(3r-1)}{\lfloor r \times 3^r}x^r$$

(II) Since general term = $(r+1)^{\text{th}}$ term = 6^{th} term $\therefore r = 5$

Hence 6^{th} term $= (-1)^5 \times \dfrac{2\cdot5\cdot8\cdot11\cdot14}{1\cdot2\cdot3\cdot4\cdot5 \times 243}x^5 = -\dfrac{308}{729}x^5$

4. (I) General term of $(1-x)^{\frac{4}{3}}$

$$= (-1)^r \times \frac{p(p-q)(p-2q)\cdots\{p-(r-1)q\}}{|\underline{r} \times q^r} x^r$$

$$= (-1)^r \times \frac{4\cdot 1\cdot(-2)\cdots(7-3r)}{|\underline{r} \times 3^r} x^r$$

(II) As before $r = 5$ $\therefore$ 6th term $= (-1)^5 \times \dfrac{4\cdot 1\cdot(-2)(-5)(-8)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 243} x^5$

$= (-1) \times - \frac{8}{729}x^5 = \frac{8}{729}x^5$

5. (I) General term of $(1+x)^{-\frac{7}{2}}$

$$= (-1)^r \times \frac{p(p+q)(p+2q)\cdots\{p+(r-1)q\}}{|\underline{r} \times q^r} x^r$$

$$= (-1)^r \times \frac{7\cdot 9\cdot 12\cdots(5+2r)}{|\underline{r} \times 2^r} x^r$$

(II) As before $r = 5$ $\therefore$ 6th term $= (-1)^5 \times \dfrac{7\cdot 9\cdot 11\cdot 13\cdot 15}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 32} x^5$

$= -\frac{9009}{256}x^5$

6. (I) General term of $(1+x)^{-\frac{8}{3}}$

$$= (-1)^r \times \frac{p(p+q)(p+2q)\cdots\{p+(r-1)q\}}{|\underline{r} \times q^r} x^r$$

$$= (-1)^r \times \frac{8\cdot 11\cdot 14\cdots(5+3r)}{|\underline{r} \times 3^r} x^r$$

(II) As before $r = 5$ $\therefore$ 6th term $= (-1)^5 \times \dfrac{8\cdot 11\cdot 14\cdot 17\cdot 20}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 243} x^5$

$= -\frac{10472}{729}x^5$

7. $(a-x)^{-1} = \{a(1-a^{-1}x)\}^{-1} = a^{-1}(1-a^{-1}x)^{-1}$

$\therefore$ (I) Gen. term of $(a-x)^{-1} = a^{-1} \times \dfrac{n(n+1)\cdots\cdots(n+r-1)}{|\underline{r}}(a^{-1}x)^r$

$= a^{-1} \times \dfrac{1\cdot 2\cdot 3\cdots r}{|\underline{r}} a^{-r}x^r = a^{-1} + a^{-r}x^r = a^{-(r+1)}x^r$

(II) $\therefore$ 6th term $= a^{-6}x^5$

8. $(a+\frac{1}{2}x)^{\frac{6}{5}} = \{a(1+\frac{1}{2}a^{-1}x)\}^{\frac{6}{5}} = a^{\frac{6}{5}}\left(1+\dfrac{x}{2a}\right)^{\frac{6}{5}}$

(I) $\therefore$ Gen. term of $(a+\frac{1}{2}x)^{\frac{6}{5}}$

$$= a^{\frac{6}{5}} \times \frac{p(p-q)(p-2q)\cdots\cdots\{p-(r-1)q\}}{|\underline{r} \times q^r}\left(\frac{x}{2a}\right)^r$$

$$= a^{\frac{6}{5}} \times \frac{6\cdot1\cdot(-4)(-9)\cdots(11-5r)}{\lfloor r \times 5^r}\left(\frac{x^r}{2^r a^r}\right)$$

$$= (-1)^r \times a^{\frac{6}{5}} \times \frac{6\cdot1\cdot4\cdots(5r-11)}{\lfloor r \times 10^r} a^{-r}x^r$$

(II) $\therefore$ 6th term $= (-1)^5 \times a^{\frac{6}{5}} \times \dfrac{6\cdot1\cdot4\cdot9\cdot14}{1\cdot2\cdot3\cdot4\cdot5\cdot10^5} a^{-5}x^5$

$$= -a^{\frac{6}{5}} \times -\tfrac{63}{250000}a^{-5}x^5 = -\tfrac{63}{250000}a^{-\frac{19}{5}}x^5$$

9. (I) Gen. term of $(1-2x)^{-2} = \dfrac{n(n+1)(n+2)\cdots(n+r-1)}{\lfloor r}(2x)^r$

$$= \frac{2\cdot3\cdot4\cdots(r+1)}{\lfloor r}2^r x^r = \frac{2\cdot3\cdot4\cdots r(r+1)}{1\cdot2\cdot3\cdots r}2^r x^r = (r+1)2^r x^r$$

(II) Since general term $= (r+1)$th term $=$ 5th term $\therefore r = 4$

Hence 5th term $= (4+1)2^4x^4 = 5 \times 16x^4 = 80x^4$

10. General term of $(1+\frac{2}{3}x^2)^{-\frac{5}{2}}$

$$= (-1)^r \times \frac{p(p+q)(p+2q)\cdots\{p+(r-1)q\}}{\lfloor r \times q^r}(\tfrac{2}{3}x^2)^r$$

$$= (-1)^r \times \frac{5\cdot7\cdot9\cdots(3+2r)}{\lfloor r \times 2^r} \times \frac{2^r x^{2r}}{3^r} = (-1)^r \times \frac{5\cdot7\cdot9\cdots(3+2r)}{\lfloor r \times 3^r}x^{2r}$$

(II) As before $r = 4$ $\therefore$ 5th term $= (-1)^4 \times \dfrac{5\cdot7\cdot9\cdot11}{1\cdot2\cdot3\cdot4\cdot81}x^8$

$$= +1 \times \tfrac{385}{216}x^8 = \tfrac{385}{216}x^8$$

11. $\left(a^{-2} + x^{-\frac{2}{3}}\right)^{-\frac{2}{5}} = \left\{a^{-2}(1 + a^2x^{-\frac{2}{3}})\right\}^{-\frac{2}{5}} = a^{\frac{4}{5}}\left(1 + a^2x^{-\frac{2}{3}}\right)^{-\frac{2}{5}}$

$\therefore$ (I) General term of $(a^{-2} + x^{-\frac{2}{3}})^{-\frac{2}{5}}$

$$= a^{\frac{4}{5}} \times (-1)^r \times \frac{p(p+q)(p+2q)\cdots\{p+(r-1)q\}}{\lfloor r \times q^r}\left(a^2x^{-\frac{2}{3}}\right)^r$$

$$= a^{\frac{4}{5}} \times (-1)^r \times \frac{2\cdot7\cdot12\cdots(5r-3)}{\lfloor r \times 5^r}a^{2r}x^{-\frac{2r}{3}}$$

$$= (-1)^r \times \frac{2\cdot7\cdot12\cdots(5r-3)}{\lfloor r \times 5^r}a^{2r+\frac{4}{5}}x^{-\frac{2r}{3}}$$

(II) $\therefore$ 5th term $= (-1)^4 \times \dfrac{2\cdot7\cdot12\cdot17}{1\cdot2\cdot3\cdot4\cdot625}a^{8+\frac{4}{5}}x^{-\frac{8}{3}} = \dfrac{119}{625}a^{\frac{44}{5}}x^{-\frac{8}{3}}$

12. $(a^{-\frac{1}{2}} - x^{-\frac{1}{2}})^{-2} = \{a^{-\frac{1}{2}}(1 - a^{\frac{1}{2}}x^{-\frac{1}{2}})\}^{-2} = a(1 - a^{\frac{1}{2}}x^{-\frac{1}{2}})^{-2}$

$\therefore$ (I) General term of $(a^{-\frac{1}{2}} - x^{-\frac{1}{2}})^{-2}$

$$= a \times \frac{n(n+1)(n+2)\cdots\cdots(n+r-1)}{\underline{|r}}(a^{\frac{1}{2}}x^{-\frac{1}{2}})^r$$

$$= a \times \frac{2\cdot3\cdot4\cdots\cdots(r+1)}{\underline{|r}}(a^{\frac{1}{2}}x^{-\frac{1}{2}})^r = a \times \frac{1\cdot2\cdot3\cdot4\cdots\cdots r\times(r+1)}{1\cdot2\cdot3\cdot4\cdots\cdots r}(a^{\frac{1}{2}}x^{-\frac{1}{2}})^r$$

$$= (r+1)a^{\frac{1}{2}r+1}x^{-\frac{r}{2}}$$

13. $r^{10} = 1024$ 14. $r^7 = 128$ 15. $0^{13} = 0$ 16. $r^{12} = 4096$

17. r is the least integer equal to or next greater than $(n+1)\frac{x}{a+x}$, or $(4+1)\frac{2}{1+2}$, or $5 \times \frac{2}{3}$ or $\frac{10}{3}$; but the first integer $> \frac{10}{3}$ is 4 $\therefore$ the greatest term of the expansion is the 4th term $= 32$

18. r is the least integer $=$ or next $> (n-1)\frac{x}{1-x}$; or $(5-1)\frac{\frac{1}{2}}{1-\frac{1}{2}}$; or 4×1; or 4 $\therefore r = 4^{\text{th}}$ term $= 5^{\text{th}}$ term $= 4\frac{3}{8}$

19. r is the least integer $=$ or next $> (n+1)\frac{x}{a+x}$; or $(20+1)\frac{3}{2+3}$; or $21 \times \frac{3}{5}$; or $\frac{63}{5}$ which is 13 $\therefore$ the greatest term is the 13^{th} term $= 125970 \times 2^8 \times 3^{12}$

20. r is the least integer $=$ or next $> (n-1)\frac{x}{1-x}$, or $(7-1)\frac{\frac{3}{5}}{1-\frac{3}{5}}$; or $6 \times \frac{3}{2}$; or 9 $\therefore$ the 9^{th} term $= \frac{19702683}{390625} =$ the 10^{th} term.

Exercise LXVI.

1. $7x < 35 \therefore x < 5$ 2. $16x - 84 > 108$, or $16x > 192 \therefore x > 12$

3. $4x < 12 \therefore x < 3$ 4. $4x + 10 > x - 20$; $3x > -30 \therefore x > -10$

5. $ax + 5bx - 5ab > a^2$; $ax - a^2 + 5bx - 5ab > 0$; $a(x-a) + 5b(x-a) > 0$; $(x-a)(a+5b) > 0 \therefore x - a > 0 \therefore x > a$. Also $bx - 7ax + 7ab < b^2$, $bx - b^2 - 7ax + 7ab < 0$, $b(x-b) - 7a(x-b) < 0$, $(b-7a)(x-b) < 0 \therefore x - b < 0 \therefore x < b$

6. $a^3 + 1 \gtreqless a^2 + a$, according $a^3 + 1 \gtreqless a(a+1)$; or as $a^2 - a + 1 \gtreqless a$; or as $a^2 + 1 \gtreqless 2a$. Now if $a = 1$, $a^2 + 1 = 2 = 2 \times 1$; but if $a > 1$ then Art. 134, $a^2 + 1 > 2a \therefore a^3 + 1 \gtreqless a^2 + a$, according as $a \gtreqless 1$

7. As above $a^3 + 1 > a^2 + a$, if $a^2 + 1 > 2a$; but Art. 134 for *all* values of a, except $a = 1$, $a^2 + 1 > 2a \therefore a^3 + 1 > a^2 + a$, when a is a negative improper fraction.

8. $\dfrac{a}{b} + \dfrac{b}{a} > 2$, if $a^2 + b^2 > 2ab$; but $a^2 + b^2 > 2ab$ by Art. 134

9. Multiplying each by 12, and reducing, we have $7x + 6 < 6x + 12$, and $7x + 6 > 6x + 10 \therefore x < 6$, and $x > 4 \therefore x = 5$

10. $a^2 + b^2 > 2ab$, Art. 134; also $a^2 + c^2 > 2ac$, and $b^2 + c^2 > 2bc$. Then by addition $a^2 + b^2 + a^2 + c^2 + b^2 + c^2 > 2ab + 2ac + 2bc$; that is $2a^2 + 2b^2 + 2c^2 > 2ab + 2ac + 2bc \therefore a^2 + b^2 + c^2 > ab + ac + bc$

11. $a^2 > a^2 - (b - c)^2$, since $(b - c)^2$ is necessarily positive $\therefore a^2 > (a - b + c)(a + b - c)$, these being the factors of $a^2 - (b - c)^2$ similarly $b^2 > (a + b - c)(b + c - a)$, and $c^2 > (a + c - b)(b + c - a)$. Multiplying unequals by unequals, $a^2b^2c^2 > (a - b + c)^2(a + b - c)^2 (b + c - a)^2$; extracting sq. root $abc > (a - b + c)(a + b - c)(b + c - a)$

13. Let $b = a + m$, and $c = a + n$, a being the least of the three quantities; then $ab(a + b) = a(a + m)(2a + m) = 2a^3 + 3a^2m + am^2$

$$ac(a + c) = a(a + n)(2a + n) = 2a^3 + 3a^2n + an^2$$

$$bc(b + c) = (a + m)(a + n)(2a + m + n)$$

$$= 2a^3 + 3a^2(m + n) + a(m + n)^2 + mn(m + n)$$

$\therefore$ by addition

(I) $ab(a + b) + ac(a + c) + bc(b + c) = 6a^3 + 6a^2(m + n) + 2a(m^2 + n^2) + 2amn + mn(m + n)$

(II) Also $6abc = 6a(a + m)(a + n) = 6a^3 + 6a^2(m + n) + 6amn$; subtracting (II) from (I) we have (I) − (II) $= 2a(m^2 - 2mn + n^2) + mn(m + n) = 2a(m - n)^2 + mn(m + n)$; but since by supposition $a < b$ and $< c$, it follows that m and n are positive quantities $\therefore 2a(m - n)^2 + mn(m + n)$ is positive $\therefore ab(a + b) + ac(a + c) + bc(b + c) - 6abc$ is a positive quantity $\therefore ab(a + b) + ac(a + c) + bc(b + c) > 6abc$

(III) Also $2(a^3 + b^3 + c^3) = 2a^3 + 2(a + m)^3 + 2(a + n)^3 = 6a^3 + 6a^2(m + n) + 6a(m^2 + n^2) + 2(m^3 + n^3)$; subtracting (I) from

(III) we have (III) – (I) $= 4a(m^2+n^2) - 2amn + 2(m^3+n^3) - mn(m+n)$
$= 4a(m^2-2mn+n^2) + 8amn - 2amn + 2(m+n)(m^2-mn+n^2) - (m+n)mn$
$= 4a(m-n)^2 + 6amn + (m+n)\{2(m^2-mn+n^2) - mn\}$
$= 4a(m-n)^2 + 6amn + (m+n)\{2(m^2-2mn+n^2) + mn\}$
$= 4a(m-n)^2 + 6amn + (m+n)\{2(m-n)^2 + mn\}$ which as before is a positive quantity $\because a$, m, and n are all positive $\therefore 2(a^3+b^3+c^3) - ab(a+b) + ac(a+c) + bc(b+c) =$ a positive quantity $\therefore ab(a+b) + ac(a+c) + bc(b+c) < 2(a^3+b^3+c^3)$

13. $3(1+a^2+a^4) - (1+a+a^2)^2 = 2 - 2a - 2a^3 + 2a^4 = 2(1-a) - 2a^3(1-a) = 2(1-a)(1-a^3)$. Now $1-a$ and $1-a^3$ have the same sign whether $a >$ or < 1 $\therefore$ their product is positive. Hence $3(1+a^2+a^4) - (1+a+a^2)^2 =$ a positive quantity $\therefore (1+a+a^2)^2 < 3(1+a^2+a^4)$ unless $a = 1$

14. $x^2y^2 - (ac+bd)^2 = (a^2+b^2)(c^2+d^2) - (ac+bd)^2 = a^2d^2 - 2abcd + b^2c^2 = (ad-bc)^2$ which is necessarily positive unless $ad = bc$; but $x^2y^2 - (ac+bd)^2 = \{xy + (ac+bd)\}\{xy - (ac+bd)\}$ $\therefore \{xy + (ac+bd)\}\{xy - (ac+bd)\} = (ad-bc)^2$ $\therefore xy - (ac+bd) = \dfrac{(ad-bc)^2}{xy+ac+bd} =$ a positive quantity $\therefore xy > ac + bd$

15. $\sqrt{a^2-b^2} + \sqrt{2ab-b^2} > a$, if $\sqrt{2ab-b^2} > a - \sqrt{a^2-b^2}$ or if $2ab - b^2 > a^2 - 2a\sqrt{a^2-b^2} + a^2 - b^2$; or if $2ab > 2a^2 - 2a\sqrt{a^2-b^2}$ or if $b > a - \sqrt{a^2-b^2}$; or if $\sqrt{a^2-b^2} > a - b$; or if $a^2 - b^2 > a^2 - 2ab + b^2$ or if $2ab > 2b^2$; or if $a > b$

16. Making the same supposition as in Ex. 13

(I) $(a+b+c)^3 = (3a+m+n)^3 = 27a^3 + 27a^2(m+n) + 9a(m+n)^2 + (m+n)^3$

(II) $27abc = 27a(a+m)(a+n) = 27a^3 + 27a^2(m+n) + 27amn$

(III) $9(a^3+b^3+c^3) = 9\{a^3 + (a+m)^3 + (a+n)^3\} = 27a^3 + 27a^2(m+n) + 27a(m^2+n^2) + 9(m^3+n^3)$

$\therefore$ (I) – (II) $= 9a(m+n)^2 - 27amn + (m+n)^3$
$= 9a(m+n)^2 - 36amn + 9amn + (m+n)^3$
$= 9a(m-n)^2 + 9amn + (m+n)^3 =$ a positive quantity

That is $(a+b+c)^3 - 27abc =$ a positive quantity

$$\therefore (a+b+c)^3 > 27abc$$

Again (III) − (I) $= 27a(m^2+n^2) - 9a(m+n)^2 + 9(m^3+n^3) - (m+n)^3$

$= 9a\{3(m^2+n^2) - (m+n)^2\} + (m+n)\{9(m^2 - mn + n^2) - (m+n)^2\}$

$= 9a(2m^2 + 2n^2 - 2mn) + (m+n)(8m^2 - 7mn + 8n^2)$

$= 18a(m^2 - mn + n^2) - 18amn + 18amn + (m+n)\{(8m^2 - 16mn + 8n^2)$
$+ 9mn\}$

$= 18a(m-n)^2 + 18amn + (m+n)\{8(m-n)^2 + 9mn\}$

$=$ a positive quantity

That is $9(a^3+b^3+c^3) - (a+b+c)^3 =$ a positive quantity

$$\therefore (a+b+c)^3 < 9(a^3+b^3+c^3)$$

17. $(a+b)(b+c)(c+a) \gtrless 8abc$, according as
$a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2 \gtrless 6abc$
or as $(ab^2 - 2abc + ac^2) + b(c^2 - 2ac + a^2) + c(a^2 - 2ab + b^2) \gtrless 0$
or as $a(b-c)^2 + b(c-a)^2 + c(a-b)^2 \gtrless 0$
But $a(b-c)^2 + b(c-a)^2 + c(a-b)^2 > 0$ unless $a = b = c$
$\therefore (a+b)(b+c)(c+a) > 8abc$

18. Let $\dfrac{x^2+34x-71}{x^2+2x-7} = m$; then $x^2+34x-71 = mx^2+2mx-7m$; that is $(m-1)x^2 + 2(m-17)x = 7m - 71$, whence $x = \dfrac{1}{m-1}\{17 - m \pm \sqrt{8(m-5)(m-9)}\}$, where if x is to be real, $m-5$ and $m-9$ must both have the same sign: i. e. m must be > 9 or < 5 $\therefore$ the given expression can have no value between 9 and 5

19. First $\dfrac{n^2-n+1}{n^2+n+1} > \dfrac{1}{3}$, if $3n^2 - 3n + 3 > n^2 + n + 1$; or if $2n^2 - 4n + 2 > 0$; or if $n^2 - 2n + 1 > 0$; or if $n^2 + 1 > 2n$; but $n^2 + 1$ is $> 2n$ $\therefore$ &c.

Secondly $\dfrac{n^2-n+1}{n^2+n+1} < 3$, if $n^2 - n + 1 < 3n^2 + 3n + 3$; or if $0 < 2n^2 + 4n + 2$; or if $0 < n^2 + 2n + 1$; or if $0 < (n+1)^2$;

but $(n+1)^2$ is necessarily positive $\therefore$ 0 *is* $< (n+1)^2$ $\therefore$ &c.

Hence $\frac{n^2-n+1}{n^2+n+1}$ lies between 3 and $\frac{1}{3}$

NOTE.—If $n = 1$, the expression $= \frac{1}{3}$.

EXERCISE LXVII.

1. $\frac{1-x^n}{1-x} = 1 + x + x^2 + x^3 + \&c. \cdots$ to n terms $= 1 + 1 + 1 + 1 + \&c. \cdots$ to n terms $= n$

2. $\frac{(x-a)(x^2+ax+a^2)}{(x-a)(x+a)} = \frac{x^2+ax+a^2}{x+a} = \frac{a^2+a^2+a^2}{a+a} = \frac{3a^2}{2a} = \frac{3a}{2}$

3. $\frac{x^{\frac{1}{2}}(x^{\frac{1}{2}} - a^{\frac{1}{2}})}{(x^{\frac{1}{2}} + a^{\frac{1}{2}})(x^{\frac{1}{2}} - a^{\frac{1}{2}})} = \frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}} + a^{\frac{1}{2}}} = \frac{a^{\frac{1}{2}}}{2a^{\frac{1}{2}}} = \frac{1}{2}$

4. $\frac{(x+7)(x-5)}{(x+3)(x-5)} = \frac{x+7}{x+3} = \frac{12}{8} = 1\frac{1}{2}$

5. $\frac{(x-\frac{1}{2})(x-3)}{(x-\frac{1}{2})(x-2)} = \frac{x+3}{x-2} = \frac{3\frac{1}{2}}{-1\frac{1}{2}} = -2\frac{1}{3}$

6. $\frac{x(x^2+b) - a(x^2+b)}{x(x-a) + b^2(x-a)} = \frac{(x-a)(x^2+b)}{(x-a)(x+b^2)} = \frac{x^2+b}{x+b^2} = \frac{a^2+b}{a+b^2}$

7. $\frac{a(x-c)(x-c)}{b(x-c)(x-c)} = \frac{a}{b}$

8. $\frac{x(a-x)}{(a-x)(a^3-a^2x-ax^2+x^3)} = \frac{a}{a^3-a^3-a^3+a^3} = \frac{a}{0} = \infty$

9. $\frac{x(x^2-a^2)+2a(x^2-a^2)}{(x-a)(x^2+ax-12a^2)} = \frac{(x+2a)(x-a)(x+a)}{(x-a)(x^2+ax-12a^2)} = \frac{(x+2a)(x+a)}{x^2+ax-12a^2}$
$= \frac{3a \times 2a}{a^2+a^2-12a^2} = \frac{6a^2}{-10a^2} = -\frac{3}{5}$

Exercise LXVIII.

1. Dividing by 3, we have $x + y + \frac{x}{3} = 3 + \frac{2}{3}$ $\therefore$ $\frac{x-2}{3}$ is integral $= t$ say. Then $x = 3t + 2$; substituting this for x in the given equation we have $3y = 11 - 4(3t + 2)$ $\therefore$ $y = 1 - 4t$; letting $t = 0$ we have $x = 2$ and $y = 1$.

2. Divide by 5, and we have $x - 2y - \frac{3y}{5} = 2 + \frac{1}{5}$ $\therefore$ $\frac{1 + 3y}{5}$ is integral $\therefore$ so also $\frac{2 + 6y}{5}$ integral $\therefore$ $\frac{2 + y}{5} = t$, whence $y = 5t - 2$. Substituting this for y in the given equation, we have $5x = 11 + 13(5t - 2)$ $\therefore$ $x = 13t - 3$. Hence taking in succession $t = 1, 2, 3$, &c., we have $x = 10, 23, 36, 49$, &c., and $y = 3, 8, 13, 18$, &c.

3. Divide by 2, and we have $x + 3y + \frac{y}{2} = 29 + \frac{1}{2}$ $\therefore$ $\frac{y-1}{2} = t$; whence $y = 2t + 1$. Substituting this for y in the given equation, we have $2x = 59 - 7(2t + 1)$ $\therefore$ $x = 26 - 7t$, and taking in succession $t = 0, 1, 2$, &c., we have $x = 26, 19, 12$ or 5, and $y = 1, 3, 5$, or 7

4. Dividing by 5, and we have $x + 2y + \frac{y}{5} = 5 + \frac{1}{5}$ $\therefore$ $\frac{y-1}{5} = t$; whence $y = 5t + 1$. Substituting this in the given equation for y, we have $5x = 26 - 11(5t + 1)$; whence $x = 3 - 11t$, and hence when $t = 0$, we have $x = 3$ and $y = 1$

5. Divide by 9, and we get $x - y - \frac{8y}{9} = \frac{2}{9}$, $\therefore$ $\frac{2 + 8y}{9}$ is integral, so also is $\frac{1 + 4y}{9}$ integral, $\therefore$ so also is $\frac{7 + 28y}{9}$ integral, $\therefore$ so also is $\frac{y + 7}{9}$ integral. Let $\frac{y + 7}{9} = t$, then $y = 9t - 7$; substituting this for y in the given equation, we have $9x = 2 + 17(9t - 7)$ $\therefore$ $x = 17t - 13$. Now writing in succession $t = 1, 2, 3$, &c., we have $x = 4, 21, 38, 55$, &c., and $y = 2, 11, 20, 29$, &c.

6. Divide by 13, and we get $x + y + \frac{8y}{13} = 6 + \frac{11}{13}$ $\therefore$ $\frac{8y - 11}{13}$ is integral, $\therefore$ $\frac{40y - 55}{13}$ is integral, $\therefore$ $3y - 4 + \frac{y - 3}{13}$ is integral, $\therefore$ $\frac{y - 3}{13} = t$, whence $y = 13t + 3$; substituting this for y in the given equation, we have $13x = 89 - 21(13t + 3)$, whence $x = 2 - 21t$. Now writing $t = 0$, we have $x = 2$ and $y = 3$

7. Divide by 12, and we get $x - 3y - \frac{5y}{12} = -1 - \frac{5}{12}$ $\therefore$ $\frac{5y - 5}{12}$ is integral, $\therefore$ so also is $\frac{y - 1}{12}$ integral. Let $\frac{y - 1}{12} = t$, then $y = 12t + 1$; substituting this in the given equation for y, we have $x = 41(12t + 1) - 17$, whence $x = 41t + 2$. Now writing in succession, 0, 1, 2, &c., for t, we have $x = 2, 43, 84, 125$, &c., and $y = 1, 13, 25, 37$, &c.

8. Divide by 37, and we get $x + y + \frac{6y}{37} = 9 + \frac{24}{37}$ $\therefore$ $\frac{6y - 24}{37}$ is integral, $\therefore$ so also is $\frac{y - 4}{37}$ which say $= t$; then $y = 37t + 4$ Then $37x = 357 - 43(37t + 4)$, whence $x = 5 - 43t$; wherefore taking $t = 0$, we have $x = 5$ and $y = 4$

9. Divide by 22, and we get $x - y - \frac{21y}{22} = \frac{6}{22}$ $\because$ $\frac{21y + 6}{22}$ is integral, $\therefore$ so also is $y - \frac{21y + 6}{22}$ integral; that is $\frac{22y - 21y - 6}{22}$, or $\frac{y - 6}{22}$ is integral $= t$, say then $y = 22t + 6$. Hence $22x = 6 + 43(22t + 6)$ $\therefore$ $x = 43t + 12$. Now writing in succession, 0, 1, 2, &c., for t we get $x = 12, 55, 98$, &c., and $y = 6, 28, 50$, &c.

10. Divide by 7, and we have $x + 3y + \frac{4y}{7} = 25 + \frac{2}{7}$ $\therefore$ $\frac{4y - 2}{7}$ is integral, $\therefore$ so also is $\frac{8y - 4}{7}$ integral, $\therefore$ $\frac{y - 4}{7} = t$, whence $y = 7t + 4$. Then $7x = 177 - 25(7t + 4)$, whence $x = 11 - 25t$. Hence taking $t = 0$, we have $x = 11$ and $y = 4$

11. Divide by 99, and we get $x - y - \frac{61y}{99} = 3 + \frac{38}{99} \therefore \frac{61y+38}{99}$ is integral, $\therefore$ so also is $\frac{305y + 190}{99}$ integral, $\therefore$ so also is $3y + 1 + \frac{8y + 91}{99}$ integral, $\therefore$ so also is $\frac{104y + 1183}{99}$ integral, $\therefore$ so also is $y + 11 + \frac{5y + 94}{99}$ integral, $\therefore$ so also is $\frac{100y + 1880}{99}$ integral, $\therefore$ so also is $y + 18 + \frac{y + 98}{99}$ integral, $\therefore y = 99t - 98$; substituting this in the given equation for y, we have $99x = 335 + 160(99t - 98)$, whence $x = 160t - 155$. Now substituting in succession 1, 2, 3, &c., for t, we have $x = 5, 165, 325, 485$, &c., and $y = 1, 100, 199, 298$, &c.

12. Divide by 4, and we have $4x - y + \frac{x}{4} = 5 + \frac{2}{4} \therefore \frac{x-2}{4} = t$, whence $x = 4t + 2$; then $4y = 17(4t + 2) - 22 \therefore y = 17t + 3$. Taking $t = 0, 1, 2, 3$, &c., we have $x = 2, 6, 10, 14$, &c., and $y = 3, 20, 37, 54$, &c.

13. Multiplying the first equation by 3, and the lower by 4, and adding the results, we have $18x + 29y = 123$. Divide by 18, and we have $x + y + \frac{11y}{18} = 6 + \frac{15}{18} \therefore \frac{11y - 15}{18}$ is integral, $\therefore$ so also is $\frac{55y - 75}{18}$ integral, $\therefore$ so also is $3y - 4 + \frac{y - 3}{18}$ integ., $\therefore y = 18t + 3$. Hence $18x = 123 - 29(18t + 3) \therefore x = 2 - 29t$. Now taking $t = 0$, we have $x = 2$, and $y = 3$, and consequently $z = 4$

14. Multiplying the upper equation by 11, the lower by 6, and adding the results, we get $56x - 49y = 469$, or $8x - 7y = 67$. Dividing this by 7, we have $x - y + \frac{x}{7} = 9 + \frac{4}{7} \therefore \frac{x-4}{7} = t$. whence $x = 7t + 4$; then $7y = 8(7t + 4) - 67$, whence $y = 8t - 5$. Now taking $t = 1, 2, 3$, &c., we have $x = 11, 18, 25$, &c., and $y = 3, 11, 19$, &c.; but since z must also be positive and integral,

we find upon trial that the only admissible values are $x = 11$, and $y = 3$, and consequently $z = 2$

15. Let x = the number of \$3 notes, and y = the number of \$5 notes; then $3x + 5y = 697$, or $x + y + \frac{2y}{3} = 232 + \frac{1}{3}$ $\therefore$ $\frac{2y-1}{3}$ is integral, $\therefore$ $\frac{4y-2}{3}$ is integral, and $\therefore$ also $\frac{y-2}{3} = t$, that is $y = 3t + 2$; then $3x = 697 - 5(3t + 2)$, whence $x = 229 - 5t$. Hence $5t < 229$, or $t < \frac{229}{5}$; i. e. $< 45\frac{4}{5}$ $\therefore$ the given sum can be made up of \$3 and \$5 notes only in 45 different ways.

16. Let x = the number of 25 cent pieces, and y = the number of 10 cent pieces; then $25x + 10y = 2730$, or $5x + 2y = 546$, $\therefore$ $2x + y + \frac{x}{2} = 273$ $\therefore$ $x = 2t$. Also $2y = 546 - 10t$ $\therefore$ $y = 273 - 5t$. Hence $5t < 273$, or $t < 54\frac{3}{5}$ $\therefore$ the given sum may be made up as directed in 54 different ways.

17. Let x = the number of guineas paid, and y = the number of half-crowns received in change; then $21x - \frac{5y}{2} = 150\frac{1}{2}$, or $42x - 5y = 301$, $\therefore$ $8x - y + \frac{2x}{5} = 60 + \frac{1}{5}$ $\therefore$ $\frac{2x-1}{5}$ is integral, $\therefore$ so also is $\frac{16x-8}{5}$ $\therefore$ $\frac{x-3}{5} = t$, or $x = 5t + 3$. Also $5y = 42(5t + 3) - 301 = 210t - 175$ $\therefore$ $y = 42t - 35$; and taking $t = 1$, we have $x = 8$, and $y = 7$

18. Let x^2 and y^2 be the two square numbers required, and assume $x^2 + y^2 = (nx - y)^2 = n^2x^2 - 2nxy + y^2$; then $x^2 = n^2x^2 - 2nxy$; or $x = n^2x - 2ny$ $\therefore$ $(n^2 - 1)x = 2ny$, or $x = \frac{2ny}{n^2-1}$, where n and y may be assumed at pleasure, and it will be found that $x^2 + y^2$ is a complete square.

But if only integral values are required assume in the expression $x = \frac{2ny}{n^2-1}$, that $y = n^2 - 1$, then $x = 2n$, where n may be

taken = any integral number, and it will be found that $x^2 + y^2$ is a complete square.

19. Let x^2 and y^2 be the two squares required, and assume $x^2 - y^2 = (x - ny)^2 = x^2 - 2nxy + n^2y^2$. Then $y^2 = 2nxy - n^2y^2$; or $y = 2nx - n^2y$; or $2nx = (n^2 + 1)y$ $\therefore$ $x = \frac{n^2 + 1}{2n} \times y$, where n and y may be assumed at pleasure, and it will be found that $x^2 - y^2$ is a complete square.

But if only integral values are required, assume in the above expression $y = 2n$; then $x = n^2 + 1$, where it will be found that when n is taken = any integral number, $x^2 - y^2$ will be a complete square.

20. Assume that the basket contains x parcels of 4 with 2 over, or y parcels of 6 with 2 over. Then $4x + 2 = 6y + 2$; or $4x - 6y = 0$; or $2x - 3y = 0$; or $x - y - \frac{y}{2} = 0$ $\therefore$ $\frac{y}{2} = t$; or $y = 2t$. Also $2x = 3y$ $\therefore$ $x = 3t$. Hence taking $t = 1, 2, 3$, &c., we have $x = 3, 6, 9, 12$, &c., and $y = 2, 4, 6, 8$, &c.

But x and y must be taken such that both $6y + 2$ and $4x + 2$ are > 90 and < 100 $\therefore$ $y = 16$, and $x = 24$, and the number of apples $= 16y + 2 = 98$

21. Let the number $= 6x + 1 = 8y + 5 = 10z + 9$

Then $6x - 8y = 4$; or $3x - 4y = 2$; or $x - y - \frac{y}{3} = \frac{2}{3}$ $\therefore$ $y = 3t - 2$, and $x = 4t - 2$

Also $6x + 1 = 10z + 9$; or $6x - 10z = 8$; or $3x - 5z = 4$, but $x = 4t - 2$ $\therefore$ $3(4t - 2) - 5z = 4$; or $12t - 5z = 10$ $\therefore$ $2t - z + \frac{2t}{5} = 2$, whence $t = 5t'$ and $z = 12t' - 2$. Then $x = 4t - 2 = 20t' - 2$; $y = 3t - 2 = 15t' - 2$, and $z = 12t' - 2$, whence taking $t' = 1$, we have $x = 18$, $y = 13$, and $z = 10$, and $\therefore$ the least number divisible as required $= 6x + 1 = (18 \times 6) + 1 = 108 + 1 = 109$

22. Let $\frac{x}{10}$ and $\frac{y}{15}$ be the two fractions, then $\frac{x}{10} + \frac{y}{15} = \frac{38}{60}$; or clearing of fractions $3x + 2y = 19$ $\therefore x + y + \frac{x}{2} = 9 + \frac{1}{2}$, and consequently $x = 2t + 1$, whence $y = 8 - 3t$. Now taking $t = 0$, 1 and 2, we have $x = 1$, 3 or 5, and $y = 8$, 5 and 2
$\therefore$ the required fractions are $\frac{1}{10}$ and $\frac{8}{15}$; $\frac{3}{10}$ and $\frac{5}{15}$; and $\frac{5}{10}$ and $\frac{2}{15}$

NOTE.—We cannot take $t = 3$, since then $y = 8 - 3t = 8 - 9 = -1 =$ a negative quantity

23. Let x, y and z = barrels respectively; then $x + y + z = 50$ (I), and $2x + 5y + 4z = 250$ (II). Multiplying (I) by 2, and subtracting the result from (II), we have $3y + 2z = 150$, whence $y = 2t$ and $z = 75 - 3t$

Also $x = 50 - y - z = 50 - 2t - (75 - 3t) = t - 25$

Then in order that z may be positive $75 - 3t$ must be positive, and $\therefore 3t < 75$, or $t < 25$, and in order that x may be positive, $t - 25$ must be positive, that is $t > 25$; therefore t is both less than and greater than 25, which is impossible.

24. Let x, y and z = the number of pieces respectively; then $x + y + z = 100$ (I), and $100x + 20y + 5z = 2000$ (II). Dividing (II) by 5, and from the result subtracting (I), we have $19x + 3y = 300$, whence $x = 3t$ and $y = 100 - 19t$ $\therefore z = 100 - x - y = 100 - 3t - (100 - 19t) = 16t$. Now taking $t = 1, 2, 3,$ &c., we have $x = 1$, 6, 9, 12 or 15; $y = 81$, 62, 43, 24 or 5; and $z = 16$, 32, 48, 64 or 80

25. $2x + 3y = 25$, whence $x = 11 - 3t$, and $y = 2t + 1$. Now taking $t = 0$, 1, 2 or 3, we have $x = 11$, 8, 5 or 2, and $y = 1$, 3, 5 or 7, and hence the parts are $2x$ and $3y = 22$ and 3; 16 and 9; 10 and 15, or 4 and 21.

26. Let x, y and z be the three parts; then $x + y + z = 24$ (I), and $36x + 24y + 8z = 516$ (II). Dividing (II) by 4, and multiplying (I) by 2, and taking the difference of the results,

we have $7x + 4y = 81$, whence $x = 4t - 1$, and $y = 22 - 7t$ $\therefore z = 24 - (4t - 1) - (22 - 7t) = 3 + 3t$. Now taking $t = 1$, 2 or 3, we have $x = 3$, 7 or 11; $y = 15$, 18 or 1; and $z = 6$, 9 or 12

27. Assume $y^n x$ to be a perfect number; then its divisors are $1, y, y^2, \ldots. y^n, x, xy, xy^2, \ldots. xy^{n-1}$ $\therefore y^n x = 1 + y + y^2 + \ldots. y^n + x + xy + xy^2 + \ldots. xy^{n-1}$. Now $1 + y + y^2 + \ldots. y^n = \dfrac{y^{n+1} - 1}{y - 1}$, and $x + xy + xy^2 + \ldots. xy^{n-1} = \dfrac{y^n - 1}{y - 1} \times x$ $\therefore y^n x = \dfrac{y^{n+1} - 1 + (y^n - 1)x}{y - 1}$; or clearing of fractions $y^{n+1}x - y^n x = y^{n+1} - 1 + y^n x - x$; or $y^{n+1}x - 2y^n x + x = y^{n+1} - 1$ $\therefore x = \dfrac{y^{n+1} - 1}{y^{n+1} - 2y^n - 1}$. Now in order that x may be a whole number, let $y^{n+1} - 2y^n = 0$, or $y = 2$; then $x = 2^{n+1} - 1$. Also let n be so assumed that $2^{n+1} - 1$ may be a prime number; then it will be found that $y^n x = 2^n \times (2^{n+1} - 1)$ will be a perfect number. Thus if $n = 2$, we have $2^2 \times (2^3 - 1) = 4 \times (8 - 1) = 4 \times 7 = 28 = 14 + 7 + 4 + 2 + 1 =$ sum of all the divisors of 28.

28. Let the number $= 10x + 7 = 12y + 9 = 14z + 11$; then $10x - 12y = 2$, or $5x - 6y = 1$, whence $x = 6t - 1$, and $y = 5t - 1$. Also $10x - 14z = 4$, or $30t - 7z = 7$, whence $t = 7t'$, and $z = 30t' - 1$. Then $x = 6t - 1 = 42t' - 1$; $y = 5t - 1 = 35t' - 1$, and $z = 30t' - 1$. Now assuming $t' = 1$, we have $x = 41$; $y = 34$; and $z = 29$. Hence the least odd integer $= 10x + 7 = 410 + 7 = 417$.

29. Let x, y and z represent the numbers respectively; then $x + y + z = 100$ (I), and $50x + 30y + 2z = 500$ (II). Dividing (II) by 2, and from the result subtracting (I), we have $24x + 14y = 150$, whence $x = 7t + 1$, and $y = 9 - 12t$ $\therefore z = 100 - (7t + 1) - (9 - 12t) = 90 + 12t$. Now t must be < 1 $\because y = 9 - 12t$ must be positive; also t must be > -1 because $x = 7t + 1$ must be positive, and since x, y and z must be integral, t can only $= 0$. Hence, when $t = 0$, we have $x = 1$, $y = 9$, and $z = 90$.

MISCELLANEOUS EXERCISES.

1. $\frac{1}{4}(1 - a) - \frac{1}{9}\{3a - 2\} = \dfrac{1}{4} - \dfrac{a}{4} - \dfrac{a}{3} + \dfrac{2}{9} = \dfrac{17 - 21a}{36}$

2. $\{(x^2 - x^{-2}) + 1\}^2 - \{(x^2 - x^{-2}) - 1\}^2$
$= (x^2 - x^{-2})^2 + 2(x^2 - x^{-2}) + 1 - \{(x^2 - x^{-2})^2 - 2(x^2 - x^{-2}) + 1\}$
$= 4(x^2 - x^{-2})$

3. The *G.C.M.* of the first three quantities is evidently $a + b$, and as it is also a measure of the remaining quantity, it is their *G.C.M.*

4. Since $x = \dfrac{b^2}{b - a}$, $x - b = \dfrac{b^2}{b - a} - b = \dfrac{b^2 - b^2 + ab}{b - a} = \dfrac{ab}{b - a}$,

and $x - a = \dfrac{b^2}{b - a} - a = \dfrac{b^2 - ab + a^2}{b - a}$

$\therefore \dfrac{x - b}{a} - \dfrac{x - a}{b} = \dfrac{b}{b - a} - \dfrac{b^2 - ab + a^2}{(b - a)b} = \dfrac{b^2 - b^2 + ab - a^2}{(b - a)b} = \dfrac{ab - a^2}{b(b - a)}$

$= \dfrac{a(b - a)}{b(b - a)} = \dfrac{a}{b}$

5. $x + y + z = 15$ (I), $x - y + z = 5$ (II), $-x - y + z = 3$ (III)
Adding (I) to (III), we have $2z = 18 \therefore z = 9$
Adding (I) to (II), we have $2x + 2z = 20 \therefore 2x = 2$, and $x = 1$
Hence $x + y + z = 1 + y + 9 = 15 \therefore y = 5$

6. $5\sqrt[3]{27 \times 5} - 3\sqrt[3]{8 \times 5} + 2\sqrt[3]{125 \times 5} - 4\sqrt[3]{64 \times 5} = 15\sqrt[3]{5} - 6\sqrt[3]{5} + 10\sqrt[3]{5} - 16\sqrt[3]{5} = (15 - 6 + 10 - 16)\sqrt[3]{5} = 3\sqrt[3]{5}$

7. $x^4 + 1 = 0$. Divide each side by x^2; then $x^2 + \dfrac{1}{x^2} = 0$
$\therefore x^2 + 2 + \dfrac{1}{x^2} = 2 \therefore x + \dfrac{1}{x} = \pm\sqrt{2}$; clearing of fractions
$x^2 \mp x\sqrt{2} = -1$; $x^2 \mp x\sqrt{2} + \left(\dfrac{\sqrt{2}}{2}\right)^2 = \frac{1}{2} - 1 = -\frac{1}{2} \therefore x \mp \dfrac{1}{\sqrt{2}} = \dfrac{\pm\sqrt{-1}}{\sqrt{2}}$
Hence $x = \dfrac{+1 + \sqrt{-1}}{\sqrt{2}}$

8. $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$; then $\frac{a+b}{b} = \frac{b+c}{c}$ $\therefore \frac{a+b}{b+c} = \frac{b}{c}$

Also $\frac{b+c}{c} = \frac{c+d}{d}$ $\therefore \frac{b+c}{c+d} = \frac{c}{d}$. But $\frac{b}{c} = \frac{c}{d}$ $\therefore \frac{a+b}{b+c} = \frac{b+c}{c+d}$

Hence $a+b : b+c :: b+c : c+d$

9. $a : c :: 2a - b : 2b - c$ $\therefore a : 2a - b :: c : 2b - c$

$\therefore \frac{a}{2a-b} = \frac{c}{2b-c}$ or $\frac{2a}{2a-b} = \frac{2c}{2b-c}$ $\therefore \frac{2a}{b} = \frac{2c}{3c-2b}$(Art. 106)

$\frac{a}{b} = \frac{c}{3c-2b}$ $\therefore bc = 3ac - 2ab$; or $3ac = bc + 2ab = b(c+2a)$

$\therefore \frac{b}{3} = \frac{ac}{c+2a} = \frac{a \times \frac{c}{2}}{a + \frac{c}{2}}$ $\therefore \frac{2b}{3} = \frac{2a \times \frac{c}{2}}{a + \frac{c}{2}}$; that is $\frac{2b}{3}$ is the *H.* mean between a and $\frac{c}{2}$ $\therefore a, \frac{2b}{3}$ and $\frac{c}{2}$ are in *H.P.*

10. Sum to n terms when r is a proper fraction

$$= \frac{a(1-r^n)}{1-r} = \frac{a\left\{1 - \left(1 - \frac{1}{p}\right)\right\}}{1 - \left(1 - \frac{1}{p}\right)^{\frac{1}{n}}} = \frac{a}{p\left\{1 - \left(1 - \frac{1}{p}\right)^{\frac{1}{n}}\right\}}$$

$$\text{Sum to } \infty = \frac{a}{1-r} = \frac{a}{1 - \left(1 - \frac{1}{p}\right)^{\frac{1}{n}}} = \frac{pa}{p\left\{1 - \left(1 - \frac{1}{p}\right)^{\frac{1}{n}}\right\}}$$

$= P$ times the sum to n terms.

11. $x^{\frac{n}{2}} - x^{-\frac{n}{2}}\big)\, x^{\frac{3n}{2}} - x^{-\frac{3n}{2}} \big(x^n + 1 + x^{-n}$

$$x^{\frac{3n}{2}} - x^{\frac{n}{2}}$$

$$x^{\frac{n}{2}} - x^{-\frac{3n}{2}}$$

$$x^{\frac{n}{2}} - x^{-\frac{n}{2}}$$

$$x^{-\frac{n}{2}} - x^{-\frac{3n}{2}}$$

$$x^{-\frac{n}{2}} - x^{-\frac{3n}{2}}$$

$$x^{\frac{2}{3}} + a^{\frac{1}{3}}x^{\frac{1}{3}} + a^{\frac{2}{3}}\big)x^{\frac{4}{3}} + a^{\frac{2}{3}}x^{\frac{2}{3}} + a^{\frac{4}{3}}\big(x^{\frac{2}{3}} - a^{\frac{1}{3}}x^{\frac{1}{3}} + a^{\frac{2}{3}}$$

$$\underline{x^{\frac{4}{3}} + a^{\frac{1}{3}}x + a^{\frac{2}{3}}x^{\frac{2}{3}}}$$

$$- a^{\frac{1}{3}}x + a^{\frac{4}{3}}$$

$$\underline{- a^{\frac{1}{3}}x - a^{\frac{2}{3}}x^{\frac{2}{3}} - ax^{\frac{1}{3}}}$$

$$a^{\frac{2}{3}}x^{\frac{2}{3}} + ax^{\frac{1}{3}} + a^{\frac{4}{3}}$$

$$\underline{a^{\frac{2}{3}}x^{\frac{2}{3}} + ax^{\frac{1}{3}} + a^{\frac{4}{3}}}$$

12. 1st trial div. $= 147x^4$

1st comp. div. $= 147x^4 - 63x^3y + 9x^2y^2$

2nd trial div. $= 147x^4 - 126x^3y + 27x^2y^2$

2nd comp. div. $= 147x^4 - 126x^3y + 111x^2y^2 - 36xy^3 + 16y^4$

13. $x^{2m-n+2n-p+2p-m} = x^{m+n+p}$; $\dfrac{a^2b^2c^2}{abc} \times x^{p-q+q-r+r-p}$
$= abc \times x^0 = abc \times 1 = abc$

14. $\{(2x^2 + \frac{1}{2}x^{-2}y^2) + y\}\{(2x^2 + \frac{1}{2}x^{-2}y^2) - y\} = (2x^2 + \frac{1}{2}x^{-2}y^2)^2 - y^2$
$= 4x^4 + 2y^2 + \frac{1}{4}x^{-4}y^4 - y^2 = 4x^4 + y^2 + \frac{1}{4}x^{-4}y^4$

$\{(x^2 + b^2) + ax\}\{(x^2 + b^2) - ax\} = (x^2 + b^2)^2 - a^2x^2 = x^4 + b^4 + 2b^2x^2 - a^2x^2$

$(x^m + y^p)(x^n + y^q) = x^{m+n} + x^my^q + x^ny^p + y^{p+q}$

15. $\dfrac{(3\sqrt{5} - 2\sqrt{3})^2}{33} + \dfrac{(2\sqrt{5} - 3\sqrt{3})(3\sqrt{5} + 2\sqrt{3})}{33}$
$= \frac{1}{33}(45 - 12\sqrt{15} + 12 + 30 - 9\sqrt{15} + 4\sqrt{15} - 18) = \frac{1}{33}(69 - 17\sqrt{15})$

16. $$\dfrac{1}{2x + \dfrac{1}{\dfrac{12x^2+1}{4x}}} = \dfrac{1}{2x + \dfrac{4x}{12x^2+1}} = \dfrac{1}{\dfrac{24x^3+6x}{12x^2+1}} = \dfrac{12x^2+1}{24x^3+6x}$$

17. $\dfrac{2x + 1 - (2x - 1)}{4(4x^2 - 1)} + \dfrac{2x + 1}{2(2x - 1)(4x^2 + 1)} = \dfrac{1}{2(4x^2 - 1)}$
$+ \dfrac{2x + 1}{2(2x - 1)(4x^2 + 1)} = \dfrac{(4x^2 + 1) + (2x + 1)(2x + 1)}{2(4x^2 - 1)(4x^2 + 1)}$
$= \dfrac{4x^2 + 1 + 4x^2 + 4x + 1}{2(4x^2 - 1)(4x^2 + 1)} = \dfrac{4x^2 + 2x + 1}{16x^4 - 1}$

18. (I) $\dfrac{x(a-c)}{(x+a)(x+c)} = \dfrac{a-c}{x+a-c} \therefore \dfrac{x}{(x+a)(x+c)} = \dfrac{1}{x+a-c}$

$\therefore x^2+ax-cx = x^2+ax+cx+ac$; $2cx = -ac \therefore x = -\dfrac{a}{2}$

(II) $\sqrt{(x-1)(x-2)} - 2 = \sqrt{(x-3)(x-4)}$; squaring $(x-1)(x-2)+4-4\sqrt{(x-1)(x-2)} = (x-3)(x-4)$; $\therefore 2\sqrt{x^2-3x+2} = 2x-3 \therefore 4x^2-12x+8 = 4x^2-12x+9 \therefore 8 = 9$ which is absurd $\therefore$ the equation has no possible roots.

(III) $\dfrac{1}{(x+3)(x-5)} + \dfrac{1}{(x-5)(x+7)} - \dfrac{1}{(x+3)(x-16)} = 0$

$\therefore (x-7)(x-16)+(x+3)(x-16)-(x-5)(x+7) = 0$

$x^2-9x-112+x^2-13x-48-x^2-2x+35 = 0$; $x^2-24x = 125$, whence $x = 12 \pm \sqrt{269}$

19. Since $n = \dfrac{b+c}{b-c}$, $\dfrac{1}{n}$ will $= \dfrac{b-c}{b+c}$; then *H.* mean between n and $\dfrac{1}{n} = \dfrac{2}{\dfrac{b+c}{b-c}+\dfrac{b-c}{b+c}} = \dfrac{2}{\dfrac{2b^2+2c^2}{b^2-c^2}} = \dfrac{b^2-c^2}{b^2+c^2}$. But $a:b::b:c$

$\therefore \dfrac{a^2}{b^2} = \dfrac{b^2}{c^2} \therefore \dfrac{a^2-b^2}{a^2+b^2} = \dfrac{b^2-c^2}{b^2+c^2} \therefore$ also $\dfrac{a^2-b^2}{a^2+b^2}$ is the *H.* mean between n and $\dfrac{1}{n}$

20. Let w = work and x, y, z = times in which *A*, *B* and *C* can separately perform it;

Then $\dfrac{w}{a}$ = *A*'s daily work + *B*'s daily work (I)

$\dfrac{w}{b}$ = *A*'s " + *C*'s " (II)

$\dfrac{w}{c}$ = *B*'s " + *C*'s " (III)

$\dfrac{w}{a} - \dfrac{w}{b}$ = *B*'s " − *C*'s " (IV)

Then adding (III) and (IV), we have $\dfrac{w}{a} - \dfrac{w}{b} - \dfrac{w}{c} - 2B$'s daily

work $= 2\dfrac{w}{y}$. Hence $\dfrac{2}{y} = \dfrac{1}{a} - \dfrac{1}{b} + \dfrac{1}{c} = \dfrac{bc - ac + ab}{abc}$

and $y = \dfrac{2abc}{bc - ac + ab}$; similarly $x = \dfrac{2abc}{ac + bc - ab}$, and $z = \dfrac{2abc}{ab + ac - bc}$

21. $\dfrac{a^2}{(a-b)(a-c)} + \dfrac{b^2}{(c-b)(a-b)} - \dfrac{c^2}{(a-c)(c-b)}$, by changing signs. Hence $l.\ c.\ m. = (a-b)(a-c)(c-b)$

$$\frac{a^2(c-b) + b^2(a-c) - c^2(a-b)}{(a-b)(a-c)(c-b)} = \frac{a^2c - a^2b + ab^2 - cb^2 - ac^2 + bc^2}{a^2c - a^2b + ab^2 - cb^2 - ac^2 + bc^2} = 1$$

22. $a^2 - \left(\dfrac{a^2 + 4b^2 - 9c^2}{4b}\right)^2 = \left(a + \dfrac{a^2 + 4b^2 - 9c^2}{4b}\right)\left(a - \dfrac{a^2 + 4b^2 - 9c^2}{4b}\right)$

$$= \frac{a^2 + 4ab + 4b^2 - 9c^2}{4b} \times \frac{9c^2 - a^2 + 4ab - 4b^2}{4b}$$

$$= \frac{(a+2b)^2 - (3c)^2}{4b} \times \frac{(3c)^2 - (a-2b)^2}{4b}$$

$$= \frac{(a + 2b + 3c)(a + 2b - 3c)}{4b} \times \frac{(3c + a - 2b)(3c - a + 2b)}{4b}$$

$$= \frac{(a + 2b + 3c)(a + 2b - 3c)(a - 2b + 3c)(2b - a + 3c)}{16b^2}$$

23. $a^4 + 2a^2b^2 + b^4 - 2a^2b^2 = (a^2+b^2)^2 - (ab\sqrt{2})^2$

$= (a^2 + ab\sqrt{2} + b^2)(a^2 - ab\sqrt{2} + b^2)$

Similarly $a^4 + 2a^2b^2 + b^4 - 3a^2b^2 = (a^2 + b^2)^2 - (ab\sqrt{3})^2$

$= (a^2 + b^2 + ab\sqrt{3})(a^2 + b^2 - ab\sqrt{3})$

24. $\dfrac{\dfrac{xy + y^2 + x^2}{y}}{\dfrac{xy + y^2 + x^2}{x}} = \dfrac{x}{y}$

25. $G.C.M.$ of $(x + 7y)(x - 4y)$, and $(x + 2y)(x - 4y)$, and $(x - y)(x - 4y)$ is $x - 4y$ $\therefore$ $l.\ c.\ m. = (x - y)(x + 2y)(x - 4y)(x + 7y)$ $= x^4 + 4x^3y - 27x^2y^2 - 34xy^3 + 56y^4$

26. When $r = +1$, the formula $S = \dfrac{a(r^n - 1)}{r - 1}$ becomes $S = \dfrac{a(1^n - 1)}{1 - 1}$ $= a(1^{n-1} + 1^{n-2} + 1^{n-3} + \text{\&c. to } n \text{ terms}) = a(1 + 1 + 1 + \text{\&c. to } n \text{ terms}) = na$

When $r = -1$, $S = \dfrac{a(r^n - 1)}{r - 1}$ becomes $S = \dfrac{a\{(-1)^n - 1\}}{-1-1}$
$= a\{(-1)^{n-1} + (-1)^{n-2} + (-1)^{n-3} + \text{\&c.}\}$
$= a(1 - 1 + 1 - 1 + 1 - 1 + \text{\&c. to } n \text{ terms}) = a$ if n is odd, and
$= a(-1 + 1 - 1 + 1 - 1 + \text{\&c.}) = 0$ if n is even.

27. (I) $a_t = a + (t-1)d$; then by same notation $a_m = a + (m-1)d$, $a_n = a + (n-1)d$, $a_p = a + (p-1)d$, and $a_q = a + (q-1)d$. Then $(p-q)(m-n)d = (m-n)(p-q)d \therefore (p-q)\{a + (m-1)d - a - (n-1)d\} = (m-n)\{a + (p-1)d - a - (q-1)d\}$; since we have merely added $a - d - a + d = 0$ to each of the 2nd factors.
$\therefore (p-q)(a_m - a_n) = (m-n)(a_p - a_q)$

(II) Since $a_t = ar^{t-1}$; therefore by same notation $a_m = ar^{m-1}$, $a_n = ar^{n-1}$, $a_p = ar^{p-1}$, and $a_q = ar^{q-1}$. Then since $(p-q)(m-n) = (m-n)(p-q)$; $r^{(p-q)(m-n)} = r^{(m-n)(p-q)} \therefore (r^{m-n})^{p-q} = (r^{p-q})^{m-n}$
But $r^{m-n} = \dfrac{r^m}{r^n}$, and $r^{p-q} = \dfrac{r^p}{r^q}$. Also since $\dfrac{r}{r} = \dfrac{r^{-1}}{r^{-1}} = 0$, multiplying by the latter, we have $\left(\dfrac{r^{m-1}}{r^{n-1}}\right)^{p-q} = \left(\dfrac{r^{p-1}}{r^{q-1}}\right)^{m-n} \therefore \left(\dfrac{ar^{m-1}}{ar^{n-1}}\right)^{p-q}$
$= \left(\dfrac{ar^{p-1}}{ar^{q-1}}\right)^{m-n}$; that is $\left(\dfrac{a_m}{a_n}\right)^{p-q} = \left(\dfrac{a_p}{a_q}\right)^{m-n}$

28. On 1st morning the watch is behind the clock by 11s., and 45 hours afterwards it is only 2s. behind $\therefore$ the watch gains upon the clock to the amount of 9s. in 45 hours, or $\frac{1}{5}$ sec. in 1 hour.

Let x = gaining rate of watch per hour; then since the gaining rate of the clock is $\frac{1}{10}$ s. in 24 h., or 1 s. in 240 h., it is $\frac{1}{240}$ s. in 1 h. $\therefore x - \frac{1}{240}$ is the gain of the watch on the clock per hour $\therefore x - \frac{1}{240} = \frac{1}{5}$, whence $x = \frac{1}{5} + \frac{1}{240} + \frac{49}{240}$; hence watch gains per day $\frac{49}{240} \times 24 = \frac{49}{10} = 4 \cdot 9$ s.

29. (I) S to 12 terms $= \dfrac{8\{(\frac{3}{2})^{12} - 1\}}{\frac{3}{2} - 1} = 16\{(\frac{3}{2})^{12} - 1\} = 2059\frac{241}{256}$

(II) 3rd term $= a + 2d = 4$, and 6th term $= a + 5d = \frac{32}{27}$
$\therefore 3d = \frac{32}{27} - 4 = -\frac{76}{27}$; hence $d = -\frac{76}{81} \therefore a = 4 - 2d = 4 + \frac{152}{81} = 5\frac{71}{81}$
$\therefore$ A series $= 5\frac{71}{81} + 4\frac{76}{81} + 4 + 3\frac{5}{81} + \text{\&c.}$

(III) 3rd term $= ar^2 = 4$, and 6th term $= ar^5 = \frac{32}{27}$ $\therefore ar^5 \div ar^2 = \frac{32}{27} \div 4$ $\therefore r^3 = \frac{8}{27}$; whence $r = \frac{2}{3}$. And $a = \dfrac{4}{r^2} = 4 \div \frac{4}{9} = 9$ $\therefore$ *G.* series $= 9 + 6 + 4 + 2\frac{2}{3} + \&c.$

30. Let x = length, then $x - 60$ = breadth in yards, and $x(x - 60) = 5500$; that is $x^2 - 60x = 5500$, whence $x = 110$, and $x - 60 = 50$

31. (I) $(x^3 - y^3)^2 \div (x - y)^2 = \left(\dfrac{x^3 - y^3}{x - y}\right)^2 = (x^2 + xy + y^2)^2$ $= x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4$

(II)

$$\begin{array}{r|rrrrrr|rrr}
1 & 7 + & 0 + & 0 + & 0 + & 5 - & 4 & + \ 0 + & 3 + & 9 \\
+0 & & + 0 + & 0 + & 0 + & 0 + & 0 & + \ 0 & & \\
-2 & & & - 14 + & 0 + & 28 - & 14 & - 66 + & 64 & \\
+1 & & & & + 7 + & 0 - & 14 & + \ 7 + & 33 - & 32 \\
\hline
& 7 + & 0 - & 14 + & 7 + & 33 - & 32 & - 59 + & 100 - & 23
\end{array}$$

$$7x^5 - 14x^3 + 7x^2 + 33x - 32 - \frac{59x^2 - 100x + 23}{x^3 + 2x - 1}$$

(III) $\dfrac{x^m - x^{-m}}{x - x^{-1}} = x^{m-1} + x^{m-3} + x^{m-5} + x^{m-7} + x^{m-9} + \&c.$

We observe here that each term is derived from that preceding it by dividing by x^2. Let us now assume that this is true to $r - 1$ terms, and we have then left as remainder $x^{m-2(r-1)} - x^{-m}$. Dividing this by $x - x^{-1}$, and we get as first term of the quotient $x^{m-2(r+1)}$ which will be the r^{th} term of the quotient of $x^m - x^{-m} \div x - x^{-1}$. But $x^{m-2r+1} = x^{m-2(r-1)+1} \div x^2 = (r-1)^{th}$ term $\div x^2$ $\therefore$ if the law is true for $r - 1$ terms, it is true for r terms. Now it evidently holds for 5 terms $\therefore$ for 6 and $\therefore$ for 7 terms and so on, and $\therefore$ it is generally true, and since the first term is x^{m-1}, and each term is derived from the preceding by $\div$ by x^2 $\therefore$ the r^{th} term is $x^{m-1-2(r-1)} = x^{m-2r+1}$. If m be an even number the quotient will contain an even number of terms, and will be $x^{m-1} + x^{m-3} + x^{m-5} + \&c. + x^{m-(m-1)} + x^{m-(m+1)} + x^{m-(m+3)} + \&c. + x^{m+1-2m} = x^m(x^{-1} + x^{-3} + x^{-5} + \&c.$ to $x^{1-m}) + x^{-1} + x^{-3} + x^{-5} + \&c.$ to x^{1-m} $\therefore$ first part of quotient $=$ second part $\times x^m$

32. (I) Let $\sqrt{37+20\sqrt{3}} = \sqrt{x}+\sqrt{y}$; then $\sqrt{37-20\sqrt{3}} = \sqrt{x}-\sqrt{y}$ $\therefore\ \sqrt{1369-1200} = \sqrt{169} = 13 = x-y$. Also $37+20\sqrt{3} = x+y+2\sqrt{xy}$ $\therefore\ x+y=37$; hence $x=25$, and $y=12$. Then $\sqrt{x}+\sqrt{y} = \sqrt{25}+\sqrt{12} = 5+2\sqrt{3}$

(II) Let $\sqrt{4x+2\sqrt{4x^2-1}} = \sqrt{x'}+\sqrt{y}$; then $\sqrt{4x-2\sqrt{4x^2-1}} = \sqrt{x'}-\sqrt{y}$ $\therefore\ \sqrt{16x^2-16x^2+4} = \sqrt{4} = 2 = x'-y$. Also $4x+2\sqrt{4x^2-1} = x'+y+2\sqrt{x'y}$ $\therefore\ x'+y=4x$ $\therefore\ 2x' = 4x+2$, or $x'=2x+1$, and $2y=4x-2$ $\therefore\ y=2x-1$. Then $\sqrt{x'}+\sqrt{y} = \sqrt{2x+1}+\sqrt{2x-1}$

33. $(a^4-x^4)^{-3} = \{a^4(1-a^{-4}x^{-4})\}^{-3} = a^{-12}(1-a^{-4}x^{-4})^{-3}$
$= a^{-12}(1+\frac{3}{1}a^{-4}x^{-4}+\frac{3\cdot4}{1\cdot2}a^{-8}x^{-8}+\frac{3\cdot4\cdot5}{1\cdot2\cdot3}a^{-12}x^{-12}+\frac{3\cdot4\cdot5\cdot6}{1\cdot2\cdot3\cdot4}a^{-16}x^{-16}$ $+\&c.)$ Hence 5th term $= a^{-12}\times\frac{3\cdot4\cdot5\cdot6}{1\cdot2\cdot3\cdot4}a^{-16}x^{-16} = a^{-12}\times15a^{-16}x^{-16}$ $= 15^{-28}x^{-16}$

34. $$C_7 = \frac{28\times27\times26\times25\times24\times23\times22}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7} = 1184040$$

35. $(x^4-4x^2+10-12x^{-2}+9x^{-4})^{\frac{1}{2}} = \{(x^4-4x^2+4)$ $+6x^{-2}(x^2-2)+9x^{-4}\}^{\frac{1}{2}} = \{(x^2-2)^2+2\times3x^{-2}(x^2-2)+(3x^{-2})^2\}^{\frac{1}{2}}$ $= x^2-2+3x^{-2}$

36. Let $x=\sqrt[3]{1}$; then $x^3=1$ and $x^3-1=0$ $\therefore\ (x-1)(x^2+x+1)=0$ $\therefore\ x-1=0$ or $x=1$. Also $x^2+x=-1$, whence $x=\frac{1}{2}(-1\pm\sqrt{-3})$ $\therefore\ \sqrt[3]{1}=1$, or $\frac{1}{2}(-1\pm\sqrt{-3})$. Also $1^2=1$, and $\{\frac{1}{2}(-1\mp\sqrt{-3})\}^2$ $= \frac{1}{2}(-1\mp\sqrt{-3})$ $\therefore\ 1^2+\{\frac{1}{2}(-1\pm\sqrt{-3})\}^2 = 1+\frac{1}{2}(-1\mp\sqrt{-3})$ $= 1+\frac{1}{2}(-1\pm\sqrt{-3})$, i. e. sum of the cube roots of unity = sum of their squares.

37. (I) $bx+ay=ab=ax-a^2+by-b^2$; $ax-bx-ay+by=a^2+b^2$, or $x(a-b)-y(a-b)=a^2+b^2$; $x=y+\frac{a^2+b^2}{a-b}$ $\therefore\ \frac{b(a^2+b^2)}{a-b}+by+ay=ab$ $\therefore\ \frac{ba^2+b^3}{a-b}+y(a+b)=ab$; $y(a+b)=ab-\frac{ba^2+b^3}{a-b} = -\frac{b^2(a+b)}{a-b}$ $= \frac{-b^2(a+b)}{a-b}$ $\therefore\ y = -\frac{b^2}{a-b} = \frac{b^2}{b-a}$, and $x=y+\frac{a^2+b^2}{a-b}$ $= \frac{b^2}{b-a}+\frac{a^2+b^2}{a-b} = \frac{-b^2+a^2+b^2}{a-b} = \frac{a^2}{a-b}$

(II) If in these equations we write x for y, we shall obtain values of x and y, which will simultaneously satisfy the given equations. Thus $x^2 = 6x + 4x$; or $x^2 - 10x = 0$ $\therefore$ $x^2 - 10x$ is a factor of the reduced equation in x. Now from first equation, $y = \dfrac{x^2 - 6x}{4}$; substitute this for y in the second equation. Then $\left(\dfrac{x^2 - 6x}{4}\right)^2 = 4x + \dfrac{3(x^2 - 6x)}{2}$ $\therefore$ $x^4 - 12x^3 + 12x^2 + 80x = 0$. But we have shown that $x^2 - 10x$ is a factor of the left hand member of this $\therefore$ $(x^2 - 10x)(x^2 - 2x - 8) = 0$. Hence $x^2 - 10x = 0$; whence $x = 0$ or 10. And $x^2 - 2x = 8$; whence $x = 4$ or -2. Then $x = y = 0$, or 10, or -2, or 4

38. Let x = yds B sold for \$1; then $x + \frac{1}{3}$ = yds A sold for \$1; $\dfrac{1}{x}$ = what B received for 1 yard, and $\dfrac{1}{x + \frac{1}{3}}$ = what A received. Then $\dfrac{90}{x} + \dfrac{40}{x + \frac{1}{3}} = 42$; whence $21x^2 - 58x = 15$ $\therefore$ $x = 3$, and $x + \frac{1}{3} = 3\frac{1}{3}$

39. Insert 5 A. means between 2 and 7; $d = \dfrac{l - a}{n - 1} = \dfrac{7 - 2}{7 - 1} = \dfrac{5}{6}$. Hence A. series is $2 + 2\frac{5}{6} + 3\frac{2}{3} + 4\frac{1}{2} + 5\frac{1}{3} + 6\frac{1}{6} + 7$; that is $2 + \frac{17}{6} + \frac{11}{3} + \frac{9}{2} + \frac{16}{3} + \frac{37}{6} + 7$. Therefore the H. series is $\frac{1}{2} + \frac{6}{17} + \frac{3}{11} + \frac{2}{9} + \frac{3}{16} + \frac{6}{37} + \frac{1}{7}$

40. The $l.\ c.\ m.$ of denominators $= (a - b)(x - a)(x - b)$; then clearing of fractions $(a + c)(x - b) - (b + c)(x - a) = (x + c)(a - b)$; or $ax + cx - ab - bc - bx - cx + ab + ac = ax + ac - bx - bc$; $ax - bc - bx + ac = ax - bc - bx + ac$. Therefore the given expression is an identity.

41. $\sqrt[4]{x^2 + 1} = 1 - \sqrt{x}$ $\therefore$ $x^2 + 1 = 1 - 4\sqrt{x} + 6x - 4x\sqrt{x} + x^2$ $\therefore$ $-4\sqrt{x} + 6x - 4x\sqrt{x} = 0$ $\therefore$ $2\sqrt{x}(2x - 3\sqrt{x} + 2) = 0$ $\therefore$ $2\sqrt{x} = 0$, whence $x = 0$

Also $2x - 3\sqrt{x} = -2$, whence $x = \left(\dfrac{3 \pm \sqrt{-7}}{4}\right)^2 = \frac{1}{8}(1 \pm 3\sqrt{-7})$

42. $ab - (ab + bc - 3ac - ab + 2ac - 2bc) = ab + bc + ac$

43. $\frac{5}{6}x^2 - \frac{2}{3}xy - \frac{3}{10}y^2 - mx + ny + \frac{3}{4}x^2 + xy - \frac{4}{15}y^2 + px - qy$
$= \frac{19}{12}x^2 + \frac{1}{3}xy - \frac{17}{30}y^2 + (p - m)x + (n - q)y$

And $\frac{5}{6}x^2 - \frac{2}{3}xy - \frac{3}{10}y^2 - mx + ny - \frac{3}{4}x^2 - xy + \frac{4}{15}y^2 - px + qy$
$= \frac{1}{12}x^2 - \frac{5}{3}xy - \frac{1}{30}y^2 - (m + p)x + (n + q)y$

44. $\dfrac{(x - 8)(x + 6)(x + 7)(x - 4)}{(x - 4)(x + 6)(x - 8)(x + 5)} = \dfrac{x + 7}{x + 5}$

45. $\dfrac{x}{(x - a)(x - b)} - \dfrac{a}{(a - b)(x - a)} + \dfrac{b}{(a - b)(x - b)} = \dfrac{1}{a - b}$

$\therefore$ *l. c. m.* of denominators $= (x - a)(x - b)(a - b)$. Hence $x(a - b) - a(x - b) + b(x - a) = (x - a)(x - b)$; that is $ax - bx + ab - ax + bx - ab = x^2 - ax - bx + ab$; that is $x^2 - (a + b)x = -ab \;\therefore\; x^2 - (a + b)x + \left(\dfrac{a + b}{2}\right)^2 = \left(\dfrac{a + b}{2}\right)^2 - ab$
$= \dfrac{a^2 - 2ab + b^2}{4} - ab = \dfrac{a^2 - 2ab + b^2}{4} \;\therefore\; x - \dfrac{a + b}{2} = \pm \dfrac{a - b}{2}$, whence $x = a$ or b

(III) Multiplying by 168

$$168 + 63x - 48x + 8 = 186 \;\therefore\; 15x = 10 \;\therefore\; x = \tfrac{2}{3}$$

46. Multiplying by $(x - y)$, $(y - z)$ and $(z - x)$ respectively, we have

$$\left.\begin{aligned} x^3 - y^3 &= 37(x - y) \\ y^3 - z^3 &= 28(y - z) \\ z^3 - x^3 &= 19(z - x) \end{aligned}\right\} \;\therefore\; \text{by addition } 18x - 9y - 9z = 0$$

$\therefore\; 2x - y = z$; substituting this in third given equation, $(2x - y)^2 + x(2x - y) + x^2 = 19 \;\therefore\; 7x^2 - 5xy + y^2 = 19$; subtract from this the first equation, and we have $6x^2 - 6xy = -18$ $\therefore\; y = \dfrac{x^2 + 3}{x}$; substitute this in the first given equation, and we have $x^2 + \left(\dfrac{x^2 + 3}{x}\right)x + \left(\dfrac{x^2 + 3}{x}\right)^2 = x^2 + x^2 + 3 + \left(\dfrac{x^2 + 3}{x}\right)^2 = 37$ clearing of fractions; $2x^4 + 9 + 9x^2 + x^4 = 37x^2 \;\therefore\; 3x^4 - 28x^2 = -9$, whence $x^2 = 9$ or $\frac{1}{3} \;\therefore\; x = \pm 3$ or $\pm \frac{1}{3}\sqrt{3}$; $y = \dfrac{x^2 + 3}{x} = \dfrac{9 + 3}{\pm 3} = \pm 4$,

or $\dfrac{\frac{1}{3}+3}{\pm\frac{1}{3}\sqrt{3}} = \dfrac{10}{\dfrac{\pm\sqrt{3}}{3}} = \dfrac{\frac{10}{3}}{\pm\sqrt{3}} = \dfrac{\pm 10\sqrt{3}}{3}$; $z = 2x - y = \pm 6 \mp 4 = \pm 2$,

or $= \pm\frac{2}{3}\sqrt{3} \mp \frac{10}{3}\sqrt{3} = \mp\frac{8}{3}\sqrt{3}$

47. The question amounts to finding the least real value of x, which will satisfy the given equation

$b^2x^2 - 2abx = m - a^2b^2 - 2a^2b - 2a^2$, where m represents the least value which makes x rational; $x^2 - \dfrac{2a}{b}x = \dfrac{m - a^2b^2 - 2a^2b - 2a^2}{b^2}$

$$x^2 - \frac{2a}{b} + \frac{a^2}{b^2} = \frac{a^2 + m - a^2b^2 - 2a^2b - 2a^2}{b^2} = \frac{m - a^2b^2 - 2a^2b - a^2}{b^2}$$

$$\therefore\ x - \frac{a}{b} = \frac{\pm\sqrt{m - a^2(b^2 + 2b + 1)}}{b} \quad \therefore\ x = \frac{a \pm \sqrt{m - a^2(b+1)^2}}{b}$$

Therefore the least value of m that will render x rational, is $m = a^2(b+1)^2$, and $\therefore$ the least possible value of the given expression is found when $x = \dfrac{a}{b}$, and is therefore $a^2(b+1)^2$

48. $\{(x^{6p} + 6 + 9x^{-6p}) - 4x^p(x^{3p} + 3x^{-3p}) + 4x^{2p}\}^{\frac{1}{2}}$
$= \{(x^{3p} + 3x^{-3p})^2 - 2 \times 2x^p(x^{3p} + 3x^{-3p}) + (2x^p)^2\}^{\frac{1}{2}}$
$= x^{3p} + 3x^{-3p} - 2x^p$

49. (I) $S = \{2a + (n-1)d\}\dfrac{n}{2} = \{6\frac{6}{7} + (8-1)\frac{20}{7}\}\frac{8}{2} = (6\frac{6}{7} + 20)4$
$= 4 \times 26\frac{6}{7} = 107\frac{3}{7}$

(II) $S = \dfrac{a(1 - r^n)}{1 - r} = \dfrac{81x^{12}\{1 - (-\frac{2}{3}x^{-2}y)^8\}}{1 - (-\frac{2}{3}x^{-2}y)}$

$$= \frac{81x^{12}\left\{1 - \left(\dfrac{256x^{-16}y^8}{243 \times 27}\right)\right\}}{1 + \frac{2}{3}x^{-2}y} = \frac{243x^{14}\left(1 - \dfrac{256x^{-16}y^8}{243 \times 27}\right)}{3x^2 + 2y}$$

$$= \frac{243x^{14} - \dfrac{256x^{-2}y^8}{27}}{3x^2 + 2y} = \frac{6561x^{14} - 256x^{-2}y^8}{81x^2 + 54y}$$

(III) S_∞ = as above $\dfrac{243x^{14}\{1 - (-\frac{2}{3}x^{-2}y)^\infty\}}{3x^2 + 2y} = \dfrac{243\{1 - (-\frac{1}{3})^\infty\}}{3 + 1}$

$$= \frac{243 \times 1}{4} = \frac{243}{4} = 60\tfrac{3}{4}$$

50. $\dfrac{a(r^3-1)}{r-1}$ = sum of first three terms, and $\dfrac{ar^3(r^6-1)}{r-1}$ = sum of next six. Then $72\left\{\dfrac{a(r^3-1)}{r-1}\right\} = \dfrac{ar^3(r^6-1)}{r-1}$, or dividing each by $\dfrac{a(r^3-1)}{r-1}$, we get $72 = r^3(r^3+1)$, whence $r^3 = 8$ or -9 $\therefore r = 2$ $\therefore$ any series having $r = 2$ will answer.

51. $x^{mn-mp+np-mn+mp-np} = x^0 = 1$

52. $$\frac{(x^4+2x^2+1)+x(x^2+1)}{(x^4+2x^2+1)-x(x^2+1)} = \frac{(x^2+1)^2+x(x^2+1)}{(x^2+1)^2-x(x^2+1)}$$
$$= \frac{(x^2+1)(x^2+x+1)}{(x^2+1)(x^2-x+1)} = \frac{x^2+x+1}{x^2-x+1}$$

53. $x^2-2(a+b)x = 3a^2-10ab+3b^2$; complete the square $x^2-2(a+b)x+(a+b)^2 = 3a^2-10ab+3b^2+a^2+2ab+b^2$ $= 4a^2-8ab+4b^2 = 4(a-b)^2$ $\therefore x-(a+b) = \pm 2(a-b)$; or $x = a+b\pm 2(a-b) = 3a-b$ or $3b-a$

54. Because $\sqrt{y-x} : \sqrt{20-x} :: 2 : 2$ $\therefore \sqrt{y-x} = \sqrt{20-x}$ $\therefore y-x = 20-x$; hence $y = 20$. Then from first given equation, $\sqrt{y}-\sqrt{20-x} = \sqrt{20-x}$ $\therefore \sqrt{y} = 2\sqrt{20-x}$, and hence $y = 80-4x$, but $y = 20$ $\therefore 20 = 80-4x$, or $4x = 60$, and $x = 15$, and $y = 20$

55. $\{(x^2+1)-2x\}\{(x^2+1)+2x\}+2(x^2+2x+1)$
$= (x^2+1)^2-4x^2+2(x^2+2x+1) = x^4+2x^2+1-4x^2+2x^2+4x+2$
$= x^4+4x+3$;

$(x^4+2x^2y^{\frac{3}{2}}+y^3)(x^4-2x^2y^{\frac{3}{2}}+y^3)-2y^3(x^4-2x^2y^{\frac{3}{2}}+y^3)$
$= (x^2+y^{\frac{3}{2}})^2(x^2-y^{\frac{3}{2}})^2+2y^3(x^4-2xy^{\frac{3}{2}}+y^3)$
$= (x^4-y^3)^2+2x^4y^3-4xy^{\frac{9}{2}}+2y^6$
$= x^8-2x^4y^3+y^6+2x^4y^3-4xy^{\frac{9}{2}}+2y^6 = x^8-4xy^{\frac{9}{2}}+3y^6$

56. Multiply 2nd given equation by 3, then $3x^2y+3xy^2 = 3b^3$. Add this to 1st given equation, and we get $x^3+3x^2y+3xy^2+y^3 = a^3+3b^3$ $\therefore x+y = \sqrt[3]{a^3+3b^3}$. But $xy(x+y) = b^3$ $\therefore xy(\sqrt[3]{a^3+3b^3}) = b^3$ $\therefore xy = \dfrac{b^3}{\sqrt[3]{a^3+3b^3}}$

Then $x^2 + 2xy + y^2 = (a^3 + 3b^3)^{\frac{2}{3}}$, and $4xy = \dfrac{4b^3}{(a^3 + 3b^3)^{\frac{1}{3}}}$

$$\therefore x^2 - 2xy + y^2 = (a^3 + 3b^3)^{\frac{2}{3}} - \frac{4b^3}{(a^3 + 3b^3)^{\frac{1}{3}}}$$

$$\text{Then } x - y = \pm\sqrt{(a^3 + 3b^3)^{\frac{2}{3}}\left(1 - \frac{4b^3}{a^3 + 3b^3}\right)}$$

$$x - y = \pm\sqrt{(a^3 + 3b^3)^{\frac{2}{3}}\left(\frac{a^3 - b^3}{a^3 + 3b^3}\right)}$$

$$x - y = \pm(a^3 + 3b^3)^{\frac{1}{3}}\sqrt{\frac{a^3 - b^3}{a^3 + 3b^3}}$$

$$x + y = (a^3 + 3b^3)^{\frac{1}{3}}$$

$$\therefore x = \tfrac{1}{2}(a^3 + 3b^3)^{\frac{1}{3}}\left(1 \pm \sqrt{\frac{a^3 - b^3}{a^3 + 3b^3}}\right)$$

$$y = \tfrac{1}{2}(a^3 + 3b^3)^{\frac{1}{3}}\left(1 \mp \sqrt{\frac{a^3 - b^3}{a^3 + 3b^3}}\right)$$

57. Let x = number at first, then $\dfrac{35}{x}$ = what each had to pay, but two left, therefore the number remainining $= x - 2$, and con- consequently $\dfrac{35}{x - 2}$ = what each paid. Hence $\dfrac{35}{x - 2} = \dfrac{35}{x} + 2$
$\therefore x^2 - 2x = 35$; or $x^2 - 2x + 1 = 36 \therefore x - 1 = \pm 6$, and $x = 7$

58. $\left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)^4 = a^2\left(1 + a^{-\frac{1}{2}}b^{\frac{1}{2}}\right)^4$

$$= a^2\left(1 + \frac{4}{1}a^{-\frac{1}{2}}b^{\frac{1}{2}} + \frac{4\cdot 3}{1\cdot 2}\left(a^{-\frac{1}{2}}b^{\frac{1}{2}}\right)^2 + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}\left(a^{\frac{1}{2}}b^{\frac{1}{2}}\right)^3 + \&c.\right.$$

Hence 4th term $= a^2 \times \dfrac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}\left(a^{-\frac{1}{2}}b^{\frac{1}{2}}\right)^3 = a^2 \times 4a^{-\frac{3}{2}}b^{\frac{3}{2}} = 4a^{\frac{1}{2}}b^{\frac{3}{2}}$

59. $(-2a^2x^4)^2 = 4a^4x^8$ $\quad -2a^2x^4 \times -3ax^7 \times 2 = 12a^3x^{11}$

$-\frac{1}{2}ax^2 \times \frac{1}{2}ax^6 \times 2 = -\frac{1}{2}a^2x^8$ $\quad -x^5 \times \frac{1}{2}ax^6 \times 2 = -ax^{11}$

$ax \times -3ax^7 \times 2 = -6a^2x^8$ $\quad \overline{(12a^3 - a)x^{11}}$

$\overline{= -\frac{13}{2}a^2x^8}$ $\quad \therefore$ coef. of $x^{11} = (12a^3 - a)$

$(4a^4 - \frac{13}{2}a^2)x^8 \therefore$ coef. of $x^8 = (4a^4 - \frac{13}{2}a^2)$

60. Let x and y be the numbers, x being the greater; then $x : y :: x + y : a$, and $x : y :: x - y : b$ $\therefore$ $y(x + y) = ax$, and $y(x - y) = bx$

$$xy + y^2 = ax$$
$$xy - y^2 = bx$$
$$2xy = ax + bx;$$ or dividing by $2x$, we have $y = \frac{1}{2}(a + b)$

$$2y^2 = ax - bx = (a - b)x \therefore x = \frac{2y^2}{a - b} = \frac{2(a + b)^2}{4(a - b)} = \frac{(a + b)^2}{2(a - b)}$$

61. (I) $$2 + \cfrac{1}{3 + \cfrac{x - 1}{4x - 3}} = 2 + \frac{4x - 3}{13x - 10} = \frac{30x - 23}{13x - 10}$$

(II) $$\left(\frac{x^2 + 1}{x^2 - 1}\right)\left(\frac{x^2 + 2 + x^{-2}}{x^3 + x^{-3}}\right) = \left(\frac{x^2 + 1}{x^2 - 1}\right)\left(\frac{(x + x^{1})^2}{x^3 + x^{-3}}\right)$$

$$= \left(\frac{x^2 + 1}{x^2 - 1}\right)\left(\frac{x + x^{-1}}{x^2 - 1 + x^{-2}}\right) = \frac{x^3 + x + x + x^{-1}}{x^4 - x^2 + 1 - x^2 + 1 - x^{-2}}$$

$$= \frac{x^3 + 2x + x^{-1}}{x^4 - 2x^2 + 2 - x^{-2}} = \frac{x^5 + 2x^3 + x}{x^6 - 2x^4 + 2x^2 - 5} = \frac{x(x^2 + 1)^2}{x^6 - 2x^4 + 2x^2 - 1}$$

62. $$\frac{x + 6}{(x + 7)(x - 5)} + \frac{x - 4}{(x + 7)(x + 3)} - \frac{x + 2}{(x - 5)(x + 3)}$$

$\therefore$ $l.\ c.\ m.$ of denom. $= (x + 7)(x - 5)(x + 3)$; then reducing to common denom. $= \dfrac{(x + 6)(x + 3) + (x - 4)(x - 5) - (x + 2)(x + 7)}{(x + 7)(x - 5)(x + 3)}$

$$= \frac{x^2 + 9x + 18 + x^2 - 9x + 20 - x^2 - 9x - 14}{x^3 + 5x^2 - 29x - 105} = \frac{x^2 - 9x + 24}{x^3 + 5x^2 - 29x - 105}$$

63. $$\left\{\left(\frac{x^2}{y^2} + 2 + \frac{y^2}{x^2}\right) - 2 \times \tfrac{1}{2}\left(\frac{x}{y} + \frac{y}{x}\right) + \left(\frac{1}{2}\right)^2\right\}^{\frac{1}{2}} = \frac{x}{y} + \frac{y}{x} - \frac{1}{2}$$

(II) $\{(x^4 - 2x^3 + x^2) + \frac{1}{2}(x^2 - x) + \frac{1}{16}\}^{\frac{1}{2}}$

$= \{(x^2 - x)^2 + 2 \times \frac{1}{4}(x^2 - x) + (\frac{1}{4})^2\}^{\frac{1}{2}} = x^2 - x + \frac{1}{4}$

64. $(x^m - 2y^n)(x^m - y^n) = x^{2m} - 2x^m y^n - x^m y^n + 2y^{2n}$
$= x^{2m} - 3x^m y^n + 2y^{2n}$;

$\{x^{m^2} + (ax^m - b)\}\{x^{m^2} - (ax^m - b)\} = (x^{m^2})^2 - (ax^m - b)^2$
$= x^{2m^2} - a^2x^{2m} + 2abx^m - b^2$

65. (I) $12(x^4 - 2^4) \div 3(x - 2) = 4(x^3 + 2x^2 + 4x + 8)$ $= 4x^3 + 8x^2 + 16x + 32$

(II) $4 - 2 + 1)20 - 22 + 11 - 3(5 - 3$ $\therefore$ quotient $= 5a^2b^3 - 3ab^4$

$$\begin{array}{r} 20 - 10 + 5 \\ \hline -12 + 6 - 3 \\ -12 + 6 - 3 \\ \hline \end{array}$$

66. Let $a : b :: b : c :: c : d$; then $a : d :: a^3 : b^3$. For $\frac{a}{d} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d}$, and $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$ $\therefore$ $\frac{a}{d} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3}$ $\therefore$ $a : d :: a^3 : b^3$

67. $x^2 < 10x - 16$, or $x^2 - 10x < -16$, or $x^2 - 10x + 25 < 9$, or $x - 5 < \pm 3$, or $x < 8$ or > 2 $\therefore$ values are 3, 4, 5, 6 and 7; or thus $(-x^2 + 10x - 16) > 0$; $-(x^2 - 10x + 16) > 0$; $-(x-2)(x-8) > 0$, $\therefore$ $x - 8$ must be negative, $\therefore$ $x < 8$ and $x - 2$ must be positive, $\therefore$ $x > 2$

68. By Art. 106, (VII) $\frac{26}{4\sqrt{x-5}} = \frac{26}{20}$ $\therefore$ $\frac{1}{\sqrt{x-5}} = \frac{1}{5}$ $\therefore$ $5 = \sqrt{x-5}$; or $x - 5 = 25$ $\therefore$ $x = 30$

69. $\frac{a}{b} + \frac{b}{a} > 2$; if $a^2 + b^2 > 2ab$, but $a^2 + b^2$ is greater than $2ab$ by Art. 134, Note 2, $\therefore$ $\frac{a}{b} + \frac{b}{a} > 2$

70. Let $n - 1$, n and $n + 1$ be the three numbers; then $(n - 1)^3 + n^3 + (n + 1)^3 =$ the sum of their cubes $= n^3 - 3n^2 + 3n - 1 + n^3 + n^3 + 3n^2 + 3n + 1 = 3n^3 + 6n = 3n(n^2 + 2)$ which is evidently divisible by $3n$, i. e. by three times the middle number.

71.
$$\begin{array}{r|rrrrr|rr} 1 & 1 + & 0 - & 4 + & 7 + & 0 & -\ 5 + & 6 \\ -5 & & -5 + & 25 - & 125 + & 690 & -\ 3950 & \\ +4 & & & +\ 4 - & 20 + & 160 & -\ 552 + & 3160 \\ \hline & 1 - & 5 + & 25 - & 138 + & 790 & -\ 4507 + & 3166 \end{array}$$

$\therefore$ quotient $= a^4 - 5a^3 + 25a^2 - 138a + 790 - \frac{4507a - 3166}{a^2 + 5a - 4}$

(II) $(x^3 - x^{-3})^2 \div (x - x^{-1})^2 = (x^2 + 1 + x^{-2})^2$
$= x^4 + 2x^2 + 3 + 2x^{-2} + x^{-4}$

72. It is evident that the $(n - 1)^{\text{th}}$ factor of $\left(x^{\frac{1}{2}} + a^{\frac{1}{2}}\right)\left(x^{\frac{1}{4}} + a^{\frac{1}{4}}\right)$ &c., that is the n^{th} factor of the given series is $x^{(\frac{1}{2})^{n-1}} + a^{(\frac{1}{2})^{n-1}}$, and that the term before this is $x^{(\frac{1}{2})^{n-2}} + a^{(\frac{1}{2})^{n-2}}$, and so on. Hence the series is the same as $\left(x^{(\frac{1}{2})^{n-1}} - a^{(\frac{1}{2})^{n-1}}\right)\left(x^{(\frac{1}{2})^{n-1}} + a^{(\frac{1}{2})^{n-1}}\right)\left(x^{(\frac{1}{2})^{n-2}} + a^{(\frac{1}{2})^{n-2}}\right)$ .... $\left(x^{\frac{1}{2}} + a^{\frac{1}{2}}\right)$. Now the product of the first two terms $= x^{(\frac{1}{2})^{n-2}} - a^{(\frac{1}{2})^{n-2}}$; and the product of this by the third factor $= x^{(\frac{1}{2})^{n-3}} - a^{(\frac{1}{2})^{n-3}}$, and so on. Hence the product of the first $n - 1$ factors will be $\left(x^{\frac{1}{2}} - a^{\frac{1}{2}}\right)$, and the product of this by the n^{th} factor $\left(x^{\frac{1}{2}} + a^{\frac{1}{2}}\right)$ will be $x - a$ which $\therefore$ = the product of the given factors.

NOTE.—All this will be perhaps more evident to the student, if he takes a numerical example, and examines how the indices are affected by multiplying them as in the above question. Thus suppose $n = 5$.

Then $\left\{x^{(\frac{1}{2})^{5-1}} - a^{(\frac{1}{2})^{5-1}}\right\}\left\{x^{(\frac{1}{2})^{5-1}} + a^{(\frac{1}{2})^{5-1}}\right\}$
$= \left(x^{(\frac{1}{2})^4} - a^{(\frac{1}{2})^4}\right)\left(x^{(\frac{1}{2})^4} + a^{(\frac{1}{2})^4}\right) = \left(x^{\frac{1}{16}} - a^{\frac{1}{16}}\right)\left(x^{\frac{1}{16}} + a^{\frac{1}{16}}\right)$
$= \left(x^{\frac{1}{8}} - a^{\frac{1}{8}}\right) = x^{(\frac{1}{2})^3} - a^{(\frac{1}{2})^3} = \left(x^{(\frac{1}{2})^{5-2}} - a^{(\frac{1}{2})^{5-2}}\right)$

73. $\dfrac{13(2x + 3) - 7(2x - 3)}{12(4x^2 - 9)} - \dfrac{x - 4}{4x^2 + 9} = \dfrac{12x + 60}{12(4x^2 - 9)} - \dfrac{x - 4}{4x^2 + 9}$

$= \dfrac{x + 5}{4x^2 - 9} - \dfrac{x - 4}{4x^2 + 9} = \dfrac{4x^3 + 20x^2 + 9x + 45 - 4x^3 + 16x^2 + 9x - 16}{16x^4 - 81}$

$= \dfrac{36x^2 + 18x + 9}{16x^4 - 81}$

74. (I) $\{(\frac{1}{4}x^2+\frac{2}{9}y^2)+\frac{1}{3}xy\}\{(\frac{1}{4}x^2+\frac{2}{9}y^2)-\frac{1}{3}xy\}=(\frac{1}{4}x^2+\frac{2}{9}y^2)^2-\frac{1}{9}x^2y^2$
$=\frac{1}{16}x^4+\frac{1}{9}x^2y^2+\frac{4}{81}y^4-\frac{1}{9}x^2y^2=\frac{1}{16}x^4+\frac{4}{81}y^4$

(II) $\left(2x^{\frac{1}{4}}+3y^{\frac{1}{4}}\right)\left(2x^{\frac{1}{4}}-3y^{\frac{1}{4}}\right)\left\{\left(4x^{\frac{1}{2}}+9y^{\frac{1}{2}}\right)+6x^{\frac{1}{4}}y^{\frac{1}{4}}\right\}$
$\left\{\left(4x^{\frac{1}{2}}+9y^{\frac{1}{2}}\right)-6x^{\frac{1}{4}}y^{\frac{1}{4}}\right\}$
$=\left(4x^{\frac{1}{2}}-9y^{\frac{1}{2}}\right)\left(16x+72x^{\frac{1}{2}}y^{\frac{1}{2}}+81y-36x^{\frac{1}{2}}y^{\frac{1}{2}}\right)$
$=\left(4x^{\frac{1}{2}}-9y^{\frac{1}{2}}\right)\left(16x+36x^{\frac{1}{2}}y^{\frac{1}{2}}+81y\right)$
$=64x^{\frac{3}{2}}+144xy^{\frac{1}{2}}+324x^{\frac{1}{2}}y-144xy^{\frac{1}{2}}-324x^{\frac{1}{2}}y-729y^{\frac{3}{2}}$
$=64x^{\frac{3}{2}}-729y^{\frac{3}{2}}$

75. Multiplying first equation by $\sqrt{2}$, and second by $\sqrt{3}$, we have $2x\sqrt{6}-6y=6\sqrt{2}$ (I); $3x\sqrt{6}-6y=5\sqrt{18}=15\sqrt{2}$ (II). Subtracting (I) from (II) $x\sqrt{6}=9\sqrt{2}$ $\therefore$ $x=\dfrac{9\sqrt{2}}{\sqrt{6}}=\dfrac{9\sqrt{12}}{6}=3\sqrt{3}$; then $2x\sqrt{3}-3y\sqrt{2}=6\sqrt{3}\times\sqrt{3}-3y\sqrt{2}=18-3y\sqrt{2}=6$ $\therefore$ $3y\sqrt{2}=12$ and $\therefore$ $y=\dfrac{12}{3\sqrt{2}}=\sqrt{2}$

76. S_n of $1+3+5$, &c., $=\{2+(n-1)2\}\dfrac{n}{2}=(2+2n-2)\dfrac{n}{2}=n^2$

S to $\frac{1}{2}n$ terms $=\{2+(\frac{1}{2}n-1)2\}\dfrac{\frac{1}{2}n}{2}=(2+n-2)\dfrac{n}{4}=\frac{1}{4}n^2$; then sum of last half of the series $=n^2-\dfrac{n^2}{4}=\frac{3}{4}n^2=3$ times $\frac{1}{4}n^2$

77. The A. mean is $\dfrac{a^2+2ab+b^2}{ab}=\dfrac{(a+b)^2}{ab}$, and the H. mean is $\dfrac{a^2-2ab+b^2}{ab}=\dfrac{(a-b)^2}{ab}$; then Art. 262, the G. mean $=\sqrt{AH}$
$=\sqrt{\dfrac{(a+b)^2(a-b)^2}{a^2b^2}}=\dfrac{(a+b)(a-b)}{ab}=\dfrac{a^2-b^2}{ab}=\dfrac{a}{b}-\dfrac{b}{a}$

78. By Art. 260, it appears that $H=\dfrac{2ac}{a+c}$ $\therefore$ $b=\dfrac{2ac}{a+c}$, substitute this for b

Then $\dfrac{a+c}{ac}=\dfrac{1}{\dfrac{2ac}{a+c}-a}+\dfrac{1}{\dfrac{2ac}{a+c}-c}=\dfrac{a+c}{a(c-a)}+\dfrac{a+c}{c(a-c)}$

$\therefore \frac{1}{ac} = \frac{1}{a(c-a)} + \frac{1}{c(a-c)} = \frac{ac - a^2 + ac - c^2}{ac(c-a)(a-c)}$

$\therefore 1 = \frac{2ac - a^2 - c^2}{(c-a)(a-c)} \therefore 2ac - a^2 - c^2 = 2ac - a^2 - c^2$. Now reversing the steps of this operation, we shall have proved the point required.

79. $v = r + s + t$, and $s = m\frac{x}{y}$, and $t = nxy^2 \therefore v = r + \frac{mx}{y} + nxy^2$

$$\left.\begin{array}{l} 0 = r + m + n \\ 8 = r + m + 27n \\ 1 = r + 0 + 0 \end{array}\right\} \text{From these equations } r = 1,\ m = -\tfrac{17}{13}, \text{ and } n = \tfrac{4}{13}$$

$\therefore v = r + \frac{mx}{y} + nxy^2 = 1 - \frac{17}{13}\frac{x}{y} + \frac{4}{13}xy^2$

80. $\{(a+b)x + (a-b)\}\{(a+b)x - (a-b)\} = (a+b)^2x^2 - (a-b)^2 = 4ab \therefore (a+b)^2x^2 = 4ab + (a-b)^2 = (a+b)^2 \therefore x^2 = 1 \therefore x = \pm 1$

(II) $\frac{ax}{b} - \frac{b}{a} = x + \frac{b}{ax} \therefore a^2x^2 - b^2x = abx^2 + b^2;\ a^2x^2 - abx^2 - b^2x = b^2;\ (a^2 - ab)x^2 - b^2x = b^2;\ x^2 - \frac{b^2}{a(a-b)}x = \frac{b^2}{a(a-b)};$

$x^2 - \frac{b^2x}{a(a-b)} + \frac{b^4}{4a^2(a-b)^2} = \frac{b^4}{4a^2(a-b)^2} + \frac{b^2}{a(a-b)}$

$= \frac{b^4}{4a^2(a-b)^2} + \frac{4ab^2(a-b)}{4a^2(a-b)^2} = \frac{b^4 + 4a^2b^2 - 4ab^3}{4a^2(a-b)^2} \therefore x - \frac{b^2}{2a(a-b)}$

$= \frac{\pm b\sqrt{b^2 + 4a^2 - 4ab}}{2a(a-b)} \therefore x = \frac{b(b \pm \sqrt{b^2 + 4a^2 - 4ab})}{2a(a-b)}$

81. Product of first two factors $= a^2 - b^2$; hence product of first three factors $= a^4 - b^4$, and product of first four factors $= a^8 - b^8$. Hence it is evident that the exponent of a or of b in first term is 2^0, in product of first two terms 2^1, in product of first three terms 2^2, of four factors 2^3, of five 2^4, and so on; hence the exponent in the product of first n factors will be 2^{n-1}, and of the series to $n+1$ factors, the exponent will be 2^n. Hence the required continued product is $a^{2n} - b^{2n}$

82.
$$\begin{array}{r|lllll}
1 & 1 & -(a+b+p) & +(ap+bp-c+q) & -(aq+bq-cp) & -qc \\
+p & & +\ p & -(ap+bp) & -cp & \\
-q & & & -\ q & +(aq+bq) & +qc \\
\hline
 & 1 & -(a+b) & -\ c & +\ 0 & +\ 0
\end{array}$$

Hence quotient $= x^2 - (a+b)x - c$

83. $\{(a^2x^6 + 2abx^4 + b^2x^2) + (2acx^2 + 2bc) + c^2x^{-2}\}^{\frac{1}{2}}$

$= \{(ax^3 + bx)^2 + 2 \times cx^{-1}(ax^3 + bx) + (cx^{-1})^2\}^{\frac{1}{2}} = ax^3 + bx + cx^{-1}$

84. $\dfrac{(x-a)(x+b)+(x+a)(x-b)}{(x^2-a^2)(x^2-b^2)} \div \dfrac{(x-a)(x-b)+(x+a)(x+b)}{(x^2-a^2)(x^2-b^2)}$

$= \dfrac{x^2-ax+bx-ab+x^2+ax-bx-ab}{(x^2-a^2)(x^2-b^2)} \times \dfrac{(x^2-a^2)(x^2-b^2)}{x^2-ax-bx+ab+x^2+ax+bx+ab}$

$= \dfrac{2x^2-2ab}{2x^2+2ab} = \dfrac{x^2-ab}{x^2+ab}$

85. $G.\ C.\ M.$ of $(x^2 + px + p^2)(x^2 - px + p^2)$, and $(x^2 + px + p^2)(x^2 + px - p^2)$ is evidently $x^2 + px + p^2$. Otherwise by ordinary rule, thus:

$x^4+p^2x^2+p^4)x^4+2px^3+p^2x^2-p^4(1$
$\quad\underline{x^4+p^2x^2+p^4}$
$\quad 2px^3-p^4 = 2p(x^3-p^3)$

$x^3-p^3)x^4+p^2x^2+p^4(x$
$\quad\underline{x^4-p^3x}$
$\quad p^2x^2+p^3x+p^4 = p^2(x^2+px+p^2)$

$x^2+px+p^2)x^3-p^3(x-p$
$\quad\underline{x^3+px^2+p^2x}$
$\quad -px^2-p^2x-p^3$
$\quad\underline{-px^2-p^2x-p^3}$

$\therefore\ G.C.M. = x^2+px+p^2$

86. $\frac{5}{2}(x+5)(x-4)$; $\frac{10}{3}(x-6)(x+5)$; and $\frac{25}{6}(x-6)(x-4)$. Hence $l.c.m. = \frac{50}{6}(x+5)(x-4)(x-6) = \frac{25}{3}(x^3-5x^2-26x+120)$

87. $x^2 - \dfrac{2(ab+1)}{a^2-1}x = \dfrac{1-b^2}{a^2-1}$; $x^2 - \dfrac{2(ab+1)}{a^2-1} + \left(\dfrac{ab+1}{a^2-1}\right)^2$

$= \left(\dfrac{ab+1}{a^2-1}\right)^2 + \dfrac{1-b^2}{a^2-1} = \dfrac{a^2b^2+2ab+1+a^2-a^2b^2-1+b^2}{(a^2-1)^2}$

$= \dfrac{a^2+2ab+b^2}{(a^2-1)^2}\ \therefore\ x - \dfrac{ab+1}{a^2-1} = \pm\dfrac{a+b}{a^2-1}$, and $x = \dfrac{ab+1 \pm (a+b)}{a^2-1}$

$= \dfrac{(ab+b)+(a+1)}{a^2-1} = \dfrac{b(a+1)+(a+1)}{(a+1)(a-1)} = \dfrac{(b+1)(a+1)}{(a-1)(a+1)} = \dfrac{b+1}{a-1}$;

or $= \dfrac{ab-b-a+1}{a^2-1} = \dfrac{(b-1)(a-1)}{(a+1)(a-1)} = \dfrac{b-1}{a+1}$

88. Multiply both numerator and denominator of the first factor by x; then

$$\frac{x^4+x^{-2}+2(x^2+1)}{x^4-x^{-2}-2(x^2-1)}\cdot\left(\frac{x^2-1}{x^2+1}\right)^2=\frac{x^{-2}(x^6+1)+2(x^2+1)}{x^{-2}(x^6-1)-2(x^2-1)}\cdot\left(\frac{x^2-1}{x^2+1}\right)^2$$

$$\left(\frac{x^{-2}(x^4-x^2+1)+2}{x^{-2}(x^4+x^2+1)-2}\right)\left(\frac{x^2-1}{x^2+1}\right)=\left(\frac{x^2-1+x^{-2}+2}{x^2+1-x^{-2}-2}\right)\left(\frac{x^2-1}{x^2+1}\right)$$

$$=\left(\frac{x^2+1+x^{-2}}{x^2-1+x^{-2}}\right)\left(\frac{x^2-1}{x^2+1}\right)=\left(\frac{x^4+x^2+1}{x^4-x^2+1}\right)\left(\frac{x^2-1}{x^2+1}\right)=\frac{x^6-1}{x^6+1}$$

89. Let n represent any square number; then $\dfrac{n-1}{2}$ will be half the next lower number, and $\dfrac{n+1}{2}$ will be half the next higher.

Then $n+\left(\dfrac{n-1}{2}\right)^2=n+\dfrac{n^2-2n+1}{4}=\dfrac{4n+n^2-2n+1}{4}=\dfrac{n^2+2n+1}{4}$

$=\left(\dfrac{n+1}{2}\right)^2$

90. Let x, y and z represent the number of hours taken by A, B and C respectively to fill or empty the cistern; consequently, in 1 hour A will fill $\dfrac{1}{x}$-th of it, B, $\dfrac{1}{y}$-th of it, and C will empty $\dfrac{1}{z}$-th of it.

$$\left.\begin{aligned}\text{Then }3\left(\frac{1}{x}-\frac{1}{z}\right)+\frac{1}{2x}&=1\\ 5\left(\frac{1}{y}-\frac{1}{z}\right)+\frac{7}{4y}&=1\\ \frac{5}{3x}+\frac{1}{2y}&=1\end{aligned}\right\}\text{i. e. once the contents of the cistern}$$

Hence $\dfrac{7}{2x}-\dfrac{3}{z}=1$; $\dfrac{27}{4y}-\dfrac{5}{z}=1$, and $\dfrac{5}{3x}+\dfrac{1}{2y}=1$.

Multiplying the first of these by $\frac{5}{3}$, and the third by $\frac{7}{2}$

$$\left.\begin{aligned}\frac{35}{6x}-\frac{5}{z}&=\frac{5}{3}\\ \frac{35}{6x}+\frac{7}{4y}&=\frac{7}{2}\end{aligned}\right\}\quad\therefore\frac{7}{4y}+\frac{5}{z}=\frac{7}{2}-\frac{5}{3}=\frac{11}{6}$$

$$\text{Also }\frac{27}{4y}-\frac{5}{z}=1$$

$$\therefore\frac{34}{4y}=\frac{17}{2y}=\frac{17}{6};\text{ whence }\frac{1}{2y}=\frac{1}{6},\text{ or }y=3$$

Then $\dfrac{27}{4y} - \dfrac{5}{z} = \dfrac{27}{12} - \dfrac{5}{z} = 1 \therefore \dfrac{5}{z} = \dfrac{27}{12} - 1 = \dfrac{15}{12}$ or $\dfrac{1}{z} = \dfrac{1}{4} \therefore z = 4$

Also $\dfrac{7}{2x} - \dfrac{3}{z} = \dfrac{7}{2x} - \dfrac{3}{4} = 1 \therefore \dfrac{7}{2x} = \dfrac{7}{4}$; or $\dfrac{1}{2x} = \dfrac{1}{4} \therefore x = 2$

91.

$$
\begin{array}{r|l}
 & 2x^5 + 2x^4 - 5x^3 + 4x^2 - 9 \\
 & \qquad 3 \\
3x^4 + 3x^3 - 10x^2 - x + 3 & 6x^5 + 6x^4 - 15x^3 + 12x^2 - 27 \;(2x \\
 & 6x^5 + 6x^4 - 20x^3 - 2x^2 + 6x \\
 & \qquad 5x^3 + 14x^2 - 6x - 27
\end{array}
$$

$$
\begin{array}{r|l}
 & 3x^4 + 3x^3 - 10x^2 - x + 3 \;(3x - 27 \\
 & 5 \\
5x^3 + 14x^2 - 6x - 27 & 15x^4 + 15x^3 - 50x^2 - 5x + 15 \\
 & 15x^4 + 42x^3 - 18x^2 - 81x \\
 & \quad - 27x^3 - 32x^2 + 76x + 15 \\
 & \qquad 5 \\
 & - 135x^3 - 160x^2 + 380x + 75 \\
 & - 135x^3 - 378x^2 + 162x + 729 \\
 & \qquad 218x^2 + 218x - 654 \\
 & \qquad = 218(x^2 + x - 3)
\end{array}
$$

$$
\begin{array}{r|l}
x^2 + x - 3 & 5x^3 + 14x^2 - 6x - 27 \;(5x + 9 \\
 & 5x^3 + 5x^2 - 15x \\
 & \qquad 9x^2 + 9x - 27 \\
 & \qquad 9x^2 + 9x - 27
\end{array}
$$

92. *l. c. m.* of $(ax + b)(px + q)$, and $(ax + b)(qx - p)$
$= (ax + b)(px + q)(qx - p)$
$= apqx^3 + (aq^2 + bpq - ap^2)x^2 - (apq + bp^2 - bq^2)x - bpq$

l. c. m. of $x(x - y)$; $\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)\left(x^{\frac{2}{3}} - y^{\frac{1}{3}}\right)$; and $y(x + y)$, that is of $x\left(x^{\frac{1}{3}} - y^{\frac{1}{3}}\right)\left(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}\right)$; $\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)\left(x^{\frac{1}{3}} - y^{\frac{1}{3}}\right)$, and $y\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)\left(x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{1}{3}}\right) = x(x - y)(x + y)y$
$= xy(x^2 - y^2) = x^3y - xy^3$

93. (I) Multiplying by 273, we have
$91x - 182a = 78x + 234a - 21x - 42a$; or $34x = 374a$, whence $x = 11a$

(II) Reducing first member, and also the second member, $\frac{x-5}{6} = \frac{x-5}{x^2-1}$; or dividing by $x - 5$, we have $\frac{1}{6} = \frac{1}{x^2 - 1}$, whence $x^2 - 1 = 6$, or $x^2 = 7$, or $x = \pm\sqrt{7}$

(III) Squaring each side, $x + 4 + 2\sqrt{2x^2 + 14x + 24} + 2x + 6 = 3x + 34 \therefore 2\sqrt{2x^2 + 14x + 24} = 24$; or $\sqrt{2x^2 + 14x + 34} = 12$ $\therefore 2x^2 + 14x + 24 = 144$, or $x^2 + 7x = 60$; $x^2 + 7x + (\frac{7}{2})^2 = 60 + \frac{49}{4} = \frac{289}{4} \therefore x + \frac{7}{2} = \pm\frac{17}{2} \therefore x = 5$ or -12

(IV) $x^2y - x^2 + 3x^2y - 3y = \sqrt{x^2 + 3y}$, but $x^2y = 5$ $\therefore 5 - x^2 + 15 - 3y = \sqrt{x^2 + 3y}$, or $20 - (x^2 + 3y) = \sqrt{x^2 + 3y}$ $\therefore (x^2 + 3y) + \sqrt{x^2 + 3y} = 20 \therefore (x^2 + 3y) + (x^2 + 3y)^{\frac{1}{2}} + \frac{1}{4} = \frac{81}{4}$ $\therefore (x^2 + 3y)^{\frac{1}{2}} = \pm\frac{9}{2} - \frac{1}{2} = 4$ or -5; squaring these, $x^2 + 3y = 16$ or 25. But since $x^2y = 5$, $x^2 = \frac{5}{y} \therefore \frac{5}{y} + 3y = 16$ or 25. Hence $3y^2 - 16y = -5$; or $3y^2 - 25y = -5$. From first of these equat. $y = 5$ or $\frac{1}{3}$. Hence $x = 1$ or $\pm\sqrt{15}$

94. $(x - 2)(x - 3)(x + 2 - \sqrt{-3})(x + 2 + \sqrt{-3}) = (x^2 - 5x + 6)(x^2 + 4x + 7) = x^4 - x^3 - 7x^2 - 9x + 42 = 0$

95. $a - a + m + a - m + m - a = m$

96. $a^8 + b^9 = a^8 + 2a^4b^4 + b^8 - 2a^4b^4 = (a^4 + b^4)^2 - (a^2b^2\sqrt{2})^2 = (a^4 + b^4 + a^2b^2\sqrt{2})(a^4 + b^4 - a^2b^2\sqrt{2})$

97. Since Art. 260, $A = \frac{1}{2}(a + b)$; $H = \frac{2ab}{a+b}$, and $G = \sqrt{ab}$, we have by substituting these values for A, H and G

$$\frac{\frac{2ab}{a+b}}{\frac{a+b}{2}} = 1 + \frac{\left(\frac{2ab}{a+b} - a\right)\left(\frac{2ab}{a+b} - b\right)}{ab}$$

$$\text{or } \frac{4ab}{(a+b)^2} = 1 + \frac{(2ab - a^2 - ab)(2ab - b^2 - ab)}{ab(a+b)^2}$$

$$\text{or } \frac{4ab}{(a+b)^2} = \frac{ab(a+b)^2 + (ab - a^2)(ab - b^2)}{ab(a+b)^2}$$

$$\therefore 4ab = \frac{ab(a+b)^2 + a(b-a)(a-b)b}{ab} = \frac{ab(a+b)^2 + ab(b-a)(a-b)}{ab}$$

$\therefore 4ab = (a+b)^2 + (b-a)(a-b) = a^2 + 2ab + b^2 + ab - b^2 - a^2 + ab$
or $4ab = 4ab$

98. Let x = minute divisions the hour hand passes over; then $12x$ = divisions passed over by minute hand. Also $60 + x$ = minute divisions passed over by minute hand between two successive transits $\therefore 12x = 60 + x$; or $11x = 60$ $\therefore x = 5\frac{5}{11}$ = minute divisions passed over by hour hand, hence time in minutes $= 5\frac{5}{11} \times 12 = 1$ h. $5\frac{5}{11}$ m.

99. Let x and y = sides of rectangle; then xy = area

$(x+a)(y-b) = xy + ay - bx - ba = xy$

$(x+c)(y-d) = xy + cy - dx - cd = xy - e$

$$\left.\begin{aligned} ay - bx &= ab \\ cy - dx &= cd - e \end{aligned}\right\} \therefore \begin{aligned} ady - bdx &= abd \\ bcy - bdx &= bcd - bc \end{aligned}$$

$\therefore (bc - ad)y = bcd - bc - abd$

whence $y = \dfrac{b(cd - e - ad)}{bc - ad}$

Also $acy - bcx = abc$

$acy - adx = acd - ae$

$(bc - ad)x = acd - ae - abc$

whence $x = \dfrac{a(cd - e - bc)}{bc - ad}$

If $ad = bc$, and $bc + e = cd$

Then $x = \dfrac{a\{cd - (bc + e)\}}{bc - ad} = \dfrac{a \times 0}{0} = \dfrac{0}{0}$ = any value whatever

Also $y = \dfrac{b(cd - e - ad)}{bc - ad} = \dfrac{b(cd - e - bc)}{bc - ad}$; since $ad = bc$

$= \dfrac{b\{cd - (e + bc)\}}{bc - ad} = \dfrac{b \times 0}{0} = \dfrac{0}{0}$

100. $\left(1 - \dfrac{a-b}{x-b}\right)^3 = 1 - \dfrac{3(a-b)}{x - 2b + a} = 1 - \dfrac{3\dfrac{a-b}{x-b}}{1 + \dfrac{a-b}{x-b}}$; for $\dfrac{a-b}{x-b}$ write y.

Then $(1-y)^3 = 1 - \dfrac{3y}{1+y}$; or $1 - 3y + 3y^2 - y^3 = 1 - \dfrac{3y}{1+y}$

$\therefore -3 + 3y - y^2 = -\dfrac{3}{1+y}$ $\therefore -3 + 3y - y^2 - 3y + 3y^2 - y^3 = -3$;
or $2y^2 - y^3 = 0$ $\therefore 2y^2 = y^3$; or $2 = y$ $\therefore \dfrac{a-b}{x-b} = 2$ $\therefore 2x - 2b = a - b$;
or $2x = a + b$, whence $x = \frac{1}{2}(a+b)$

101.

$$\begin{array}{r|rrrrrr|rrrr}
1 & 5 + & 0 + & 0 - & 3 + & 0 + & 1 & + \quad 0 + & 0 + & 0 + & 0 \\
+2 & & +10 & +20 & +10 & -46 & & -122 & -104 & +158 & +628 \\
-3 & & & -15 & -30 & -15 & +69 & +183 & +156 & -237 & -942 \\
\hline
 & 5 + & 10 + & 5 - & 23 & -61 & +70 & & & & \\
 & & & & & -61 & -52 & +79 & +314 & +391 &
\end{array}$$

$$5x^3 + 10x^2 + 5x - 23 - \frac{61x - 70}{x^2 - 2x + 3}$$

or $5x^3 + 10x^2 + 5x - 23x^0 - 61x^{-1} - 52x^{-2} + 79x^{-3} + 314x^{-4} + 391x^{-5} + \&c.$

102. $G.C.M.$ of $(y-3)x^2 + (y^2-9)x - y(2y^2 - 3y - 9)$
and $(y+1)x^2 + 2(y+1)^2x - y(3y^2 + 5y + 2)$
$= G.C.M.$ of $(y-3)x^2 + (y-3)(y+3)x - y(y-3)(2y+3)$
and $(y+1)x^2 + 2(y+1)^2x - y(y+1)(3y+2)$
$= G.C.M.$ of $x^2 + (y+3)x - y(2y+3)$ and $x^2 + 2(y+1)x - y(3y+2)$
$= G.C.M.$ of $(x-y)(x+2y+3)$ and $(x-y)(x+3y+2)$
$= x - y$. See Algebra Art. 73

If the student does not clearly understand this method of factoring, he may obtain the $G.C.M.$ by rule.

Thus $x^2 + (y+3)x - y(2y+3)\big)\,x^2 + 2(y+1)x - y(3y+2)\,\big(1$

$$\begin{array}{r}
x^2 + (y+3)x - y(2y+3) \\
\hline
(y-1)x - y(y-1) \\
= (y-1)(x-y)
\end{array}$$

Then $x - y\big)\,x^2 + (y+3)x - y(2y+3)\,\big(x + (2y+3)$

$$\begin{array}{l}
x^2 - xy \\
\hline
(2y+3)x - y(2y+3) \\
(2y+3)x - y(2y+3) \\
\hline
\end{array}$$

Hence $G.C.M. = x - y$

When $y = 1$, the given quantities become $-2x^2 - 8x + 10$, and $2x^2 + 8x - 10$, of which the *G.C.M.* is evidently $x^2 + 4x - 5$

103. $a = mb^{\frac{1}{2}}$, and $c^2 = nb^3 \therefore c = n^{\frac{1}{2}}b^{\frac{3}{2}} \therefore ac = mn^{\frac{1}{2}}b^2 \therefore ac \propto b^2$

104. $(a^4)^3 + (m^4)^3 = (a^4 + m^4)(a^8 - a^4m^4 + m^8) = (a^2 + am\sqrt{2} + m^2)$ $(a^2 - am\sqrt{2} + m^2)(a^4 + a^2m^2\sqrt{3} + m^4)(a^4 - a^2m^2\sqrt{3} + m^4)$. See problem 23 in Miscellaneous Exercises.

105. $\dfrac{m^2 - (p - q)^2}{(m + q)^2 - p^2} = \dfrac{\{m + (p - q)\}\{m - (p - q)\}}{\{(m + q) - p\}\{(m + q) + p\}}$

$= \dfrac{(m + p - q)(m - p + q)}{(m - p + q)(m + p + q)} = \dfrac{m + p - q}{m + p + q}$

$\dfrac{p^2 - (q - m)^2}{(m + p)^2 - q^2} = \dfrac{(p - q + m)(p + q - m)}{(m + p - q)(m + p + q)} = \dfrac{p + q - m}{m + p + q}$

$\dfrac{q^2 - (m - p)^2}{(p + q)^2 - m^2} = \dfrac{(q - m + p)(q + m - p)}{(p + q + m)(p + q - m)} = \dfrac{q + m - p}{p + q + m}$

$\therefore$ sum of three fractions $= \dfrac{m + p - q}{m + p + q} + \dfrac{p + q - m}{m + p + q} + \dfrac{q + m - p}{p + q + m}$

$= \dfrac{m + p - q + p + q - m + q + m - p}{an + p + q} = \dfrac{m + p + q}{m + p + q} = 1$

106. $2 \times 2^x + 2^{2x} = 80 \therefore 2^{2x} + 2(2^x) + 1 = 81 \therefore 2^x + 1 = 9$ $2^x = 8 = 2^3 \therefore x = 3$

107. Let $x^2 - 8x + 22 = m$, then $x^2 - 8x + 16 = m - 6$; or $x - 4 = \pm\sqrt{m - 6}$, and $x = 4 \pm \sqrt{m - 6}$, in which $m - 6$ cannot be negative if x be real, that is m must not be less than 6

108. If a, b, c had each $\propto d$; then $a = md$, $b = nd$ and $c = pd$ $\therefore abc = mnpd^3$, that is $abc \propto d^3$

Now by hypothesis $a \propto d^2 \therefore a = md^2 \therefore a^3 = m^3d^6$

$$\text{Also } b^3 \propto nd^4 \therefore b^3 = nd^4$$

$$\text{And } c^3 \propto \frac{1}{d} \therefore c^3 = pd^{-1}$$

$$\therefore a^3b^3c^3 = m^3npd^9$$

Then taking cube root of each side, $abc = \sqrt[3]{m^3np} \times d^3$ $\therefore abc \propto d^3 \propto$ as if each of the three, a, b, c, varied directly as d

109. S_n of $1 + 3 + 5 + \&c. = n^2$. (See Ex. LIX, Example 13).

S_n of $(2m+1) + (2m+3) + (2m+5) + \&c. = \{2(2m+1) + (n-1)2\}\frac{n}{2}$

$= (4m + 2 + 2n - 2)\frac{n}{2} = (4m + 2n)\frac{n}{2} = (2m + n)n = 2mn + n^2,$

and it is manifest that the latter sum exceeds the former by $2mn$, i. e. by twice the product of m and n

110. Let β and γ be the roots, then $\beta : \gamma :: m : n$ $\therefore \frac{\beta}{\gamma} = \frac{m}{n}$

And Art. 208 *Cor.*, $\beta + \gamma = -\frac{b}{a}$, and $\beta\gamma = \frac{c}{a}$

Then Art. 106, $\frac{\beta + \gamma}{\gamma} = \frac{m+n}{n}$, $\therefore \frac{-\frac{b}{a}}{\gamma} = \frac{m+n}{n}$

$\therefore \gamma = -\frac{b}{a} \times \frac{n}{m+n}$, and $\beta = \frac{m}{n} \times \gamma = \frac{m}{n} \times -\frac{b}{a} \times \frac{n}{m+n} = -\frac{b}{a} \times \frac{m}{m+n}$

$\therefore \beta\gamma = \left(-\frac{b}{a} \times \frac{n}{m+n}\right) \times \left(-\frac{b}{a} \times \frac{m}{m+n}\right) = \frac{b^2}{a^2} \times \frac{mn}{(m+n)^2}$. But $\beta\gamma = \frac{c}{a}$

$\therefore \frac{c}{a} = \frac{b^2}{a^2} \cdot \frac{mn}{(m+n)^2}$; or $\frac{a^2c}{ab^2} = \frac{mn}{(m-n)^2}$ $\therefore \frac{b^2}{ac} = \frac{(m+n)^2}{mn}$

111. The denominator $= a^2(b - c) + b^2c - bc^2 - b^2a + ac^2$

$= a^2(b - c) + bc(b - c) - a(b^2 - c^2)$

$= (b - c)\{a^2 - a(b + c) + bc\}$

$= (b - c)(a - b)(a - c)$

Similarly the numerator $= a^4(b^2 - c^2) + b^4c^2 - b^2c^4 - a^2b^4 + a^2c^4$

$= a^4(b^2 - c^2) + b^2c^2(b^2 - c^2) - a^2(b^4 - c^4)$

$= (b^2 - c^2)\{a^4 - a^2(b^2 + c^2) + b^2c^2\}$

$= (b^2 - c^2)(a^2 - b^2)(a^2 - c^2)$

Then $\frac{(b^2 - c^2)(a^2 - b^2)(a^2 - c^2)}{(b - c)(a - b)(a - c)} = (a + b)(b + c)(a + c)$

112. (II) Every number is of the form of $3n$ or $3n \pm 1$ $\therefore$ every square number is of the form of $9n^2$ or $9n^2 \pm 6n + 1$, the former is evidently divisible by 3, and the latter becomes so when increased by 2.

(I) Of any three consecutive integers, *one* must be divisible by 3, and since every even integer is divisible by 2, and two of the given integers are even, it is manifest the latter of the two even integers is divisible by 4, and hence that the product of the three given integers must be divisible by $3 \times 2 \times 4 = 24$

113. $\{n(n+1)\}^2 - \{n(n-1)\}^2 = n^2\{(n+1)^2 - (n-1)^2\}$
$= n^2(n+1-n+1)(n+1+n-1) = n^2 \times 2 \times 2n = 4n^3$

114. $x = \dfrac{1+a}{1-a}$ $\therefore$ $x + 1 = \dfrac{2}{1-a}$, and $x^2 + 1 = \dfrac{2(1+a^2)}{(1-a)^2}$

$y = \dfrac{1+b}{1-b}$ $\therefore$ $y + 1 = \dfrac{2}{1-b}$, and $xy = \dfrac{(1+a)(1+b)}{(1-a)(1-b)}$ $\therefore$ $xy + 1 = \dfrac{2(1+ab)}{(1-a)(1-b)}$

$$\text{Then } \frac{(ab+1)(x^2+1)}{(xy+1)(a^2+1)} - \frac{x+1}{y+1} = \frac{(ab+1)\left\{\dfrac{2(1+a^2)}{(1-a)^2}\right\}}{\left\{\dfrac{2(1+ab)}{(1-a)(1-b)}\right\}(a^2+1)} - \frac{\dfrac{2}{1-a}}{\dfrac{2}{1-b}}$$

$$= \frac{\dfrac{2(ab+1)(1+a^2)}{(1-a)^2}}{\dfrac{2(1+ab)(a^2+1)}{(1-a)(1-b)}} - \frac{1-b}{1-a} = \frac{(1-a)(1-b)}{(1-a)^2} - \frac{1-b}{1-a}$$

$$= \frac{1-b}{1-a} - \frac{1-b}{1-a} = 0$$

115.
$$\begin{array}{r|rrrrrrr}
1 & 7 & +21 & +35 & +35 & +21 & +7 \\
-1 & & -7 & -14 & -21 & -14 & -7 \\
\hline
 & 7 & +14 & +21 & +14 & +7 &
\end{array} = 7x^4 + 14x^3y + 21x^2y^2 + 14xy^3 + 7y^4$$

$$\begin{array}{r|rrrrr}
1 & 7 & +14 & +21 & +14 & +7 \\
-1 & & -7 & -7 & -7 & \\
-1 & & & -7 & -7 & -7 \\
\hline
 & 7 & +7 & +7 & &
\end{array} = 7x^2 + 7xy + 7y^2$$

116.
$$
\begin{array}{rl}
8+4-6-1+1\,\big) & 72+36-68-16+16\,\big(9 \\
 & \underline{72+36-54-9+9} \\
 & \qquad\quad -14-7+7 \\
 & \qquad -7\big)\;\; 2+1-1
\end{array}
$$

$$
\begin{array}{rl}
2+1-1\,\big) & 8+4-6-1+1\,\big(4+0-1 \\
 & \underline{8+4-4} \\
 & \underline{+0-2-1+1} \\
 & \underline{\quad -2-1+1}
\end{array}
\quad \therefore\ G.C.M. = 2x^2+x-1
$$

117. $(2x^2+x-1)(9x^2-4) = (2x-1)(x+1)(3x+2)(3x-2)$
$(2x^2+x-1)(4x^2-1) = (2x-1)(x+1)(2x+1)(2x-1)$

118. *l.c.m.* of denominators $= 8(1+x^2)(1+x)(1-x)^2$
$\therefore$ the given expression =

$$\frac{6(1+x)(1+x^2)+3(1-x)(1+x)(1+x^2)+(1-x)^2(1+x^2)-2(1-x)^3(1+x)}{8(1+x)(1+x^2)(1-x)^2}$$

$$= \frac{8+8x+8x^2}{8(1-x^2)(1+x^2)(1-x)} = \frac{1+x+x^2}{(1-x^4)(1-x)} = \frac{1+x+x^2}{1-x-x^4+x^5}$$

119. $\left(x+\frac{5a}{2}\right)\left(x-\frac{3a}{2}\right) + ax = (x+5a)(x-3a) + 11\frac{1}{4}$

$\therefore x^2+2ax-\frac{15a}{4} = x^2+2ax-15a+11\frac{1}{4}$; $2ax-2ax = 15a-\frac{15a}{4}-11\frac{1}{4}$

$\therefore\ x(2a-2a) = \frac{45a-45}{4}\quad \therefore\ x = \frac{45a-45}{4(2a-2a)} = \frac{45a-45}{0}$

$\therefore$ it is an indeterminate equation. If $a = 1$ it becomes an identity.

120. (1) If a, b, c are in H. Prog., $\frac{1}{a}$, $\frac{1}{b}$ and $\frac{1}{c}$ are in A. Prog.

$\therefore\ \frac{1}{b}-\frac{1}{a} = \frac{1}{c}-\frac{1}{b}$; but $\frac{1}{b}-\frac{1}{a} = \left(\frac{1}{b}+\frac{1}{c}\right)-\left(\frac{1}{a}+\frac{1}{c}\right)$
$= \frac{b+c}{bc}-\frac{a+c}{ac}$

Also $\frac{1}{c}-\frac{1}{b} = \left(\frac{1}{c}+\frac{1}{a}\right)-\left(\frac{1}{b}+\frac{1}{c}\right) = \frac{a+c}{ac}-\frac{a+b}{ab}$

$\therefore \dfrac{a+c}{ac} - \dfrac{a+b}{ab} = \dfrac{b+c}{bc} - \dfrac{a+c}{ac} \therefore \dfrac{a+b}{ab}, \dfrac{a+c}{ac}$ and $\dfrac{b+c}{bc}$ are in A. P.

And $\dfrac{ab}{a+b}, \dfrac{ac}{a+c}$ and $\dfrac{bc}{b+c}$ are in H. Prog.

(II) Since $\dfrac{1}{b} - \dfrac{1}{a} = \dfrac{1}{c} - \dfrac{1}{b}$, we have multiplying by $a+b+c$

$$\frac{a+b+c}{b} - \frac{a+b+c}{a} = \frac{a+b+c}{c} - \frac{a+b+c}{b}$$

$$\therefore \frac{a+c}{b} + 1 - \frac{b+c}{a} - 1 = \frac{a+b}{c} + 1 - \frac{a+c}{b} - 1$$

$$\therefore \frac{a+c}{b} - \frac{b+c}{a} = \frac{a+b}{c} - \frac{a+c}{b}$$

$\therefore \dfrac{b+c}{a}, \dfrac{a+c}{b}$ and $\dfrac{a+b}{c}$ are in A. Prog.

121. $A \propto b$ and $c \propto d$ $\therefore a = mb$, and $c = nd$, $\therefore d = \dfrac{c}{n}$

Then $ad = \dfrac{m}{n}bc \therefore ad \propto bc$

122. Area of circle varies as $(\text{radius})^2 \therefore$ area $= m(\text{radius})^2$ $\therefore$ area of circles $= m9 + m9 + m16 + m25 + m36 + m49$ $=$ by addition to $m144 = m12^2$ $\therefore$ radius of resulting circle $= 12$

123. $S = \{2a + (n-1)d\}\dfrac{n}{2} = \{22 + (3-1)d\}\dfrac{3}{2}$ = sum of 1st 3 terms

$= \{22 + (9-1)d\}\dfrac{9}{2}$ = sum of 1st 9 terms

$\therefore (11+d)3 = (11+4d)9$; $11 + d = 33 + 12d \therefore 11d = -22$ or $d = -2 \therefore$ series is 11, 9, 7, 5, &c.

124. Let m = the m^{th}, and n = the n^{th} terms of a G. series; also let a = 1st term, and r = common ratio, then $m = ar^{m-1}$ and $n = ar^{n-1}$

$$\therefore \frac{m}{n} = \frac{ar^{m-1}}{ar^{n-1}} = \frac{r^{m-1}}{r^{n-1}} = r^{m-1-n+1} = r^{m-n} \quad \therefore r = \left(\frac{m}{n}\right)^{\frac{1}{m-n}};$$

$$a = \frac{m}{r^{m-1}} = \frac{m}{\left(\frac{m}{n}\right)^{\frac{m-1}{m-n}}} = m^{1-\frac{m-1}{m-n}} n^{\frac{m-1}{m-n}} = m^{\frac{-(n-1)}{m-n}} n^{\frac{m-1}{m-n}} = \frac{n^{\frac{m-1}{m-n}}}{m^{\frac{n-1}{m-n}}}$$

$= \left(\frac{n^{m-1}}{m^{n-1}}\right)^{\frac{1}{m-n}}$ $\therefore$ first term $= \left(\frac{n^{m-1}}{m^{n-1}}\right)^{\frac{1}{m-n}}$, and common ratio $= \left(\frac{m}{n}\right)^{\frac{1}{m-n}}$, where $m =$ the m^{th} and n the n^{th} terms of the series.

If one of the terms be taken as the first, the above becomes $r = \left(\frac{n}{a}\right)^{\frac{1}{n-1}}$

125. $a = 3$; $ar^4 = \frac{16}{27}$ $\therefore r^4 = \frac{16}{81}$ $\therefore r = \pm \frac{2}{3}$. Art. 254, $r = \frac{s-a}{s-l}$ $= \frac{2\frac{1}{27} - 3}{2\frac{1}{27} - \frac{16}{27}} = \frac{-\frac{26}{27}}{\frac{39}{27}} = -\frac{2}{3}$ $\therefore$ the series is $3 - 2 + \frac{4}{3} - \frac{8}{9} + \frac{16}{27}$

126. $S_n = \{2a + (n-1)d\}\frac{n}{2}$; $S_{2n} = \{2a + (2n-1)d\}n$ and $S_{3n} = \{2a + (3n-1)d\}\frac{3n}{2}$. Then latter half of $S_{2n} = S_{2n} - S_n = \{2a + (2n-1)d\}n - \{2a + (n-1)d\}\frac{n}{2}$ $= an + \frac{3}{2}n^2d - \frac{nd}{2} = an + (3n-1)\frac{nd}{2} = \{2a + (3n-1)d\}\frac{n}{2}$ $= \frac{1}{3}(2a + (3n-1)d\}\frac{3n}{2}$

127. Since S_1 and S_2 are each to n terms, we have $S_1 + S_2 = 1 + 5 + 9 +$ &c. to n terms $+ 3 + 7 + 11 +$ &c. to n terms $= 1 + 3 + 5 + 7 + 9 + 11 +$ &c. to $2n$ terms $= (2n)^2 = 4n^2$. Also $S_1 - S_2 = \{(1-3) + (5-7) + (9-11) +$ &c. to n terms$\}$ $\therefore (S - S_2)^2 = \{(1-3) + (5-7) +$ &c. to n terms$\}^2 = (-2-2-2-$ &c. to n terms$)^2 = (-2n)^2 = 4n^2$ $\therefore S_1 + S_2 = (S_1 - S_2)^2$

When $S_2 =$ sum to $n - 1$ terms

$S_1 + S_2 = 1 + 3 + 5 + 7 +$ &c. to $(2n-1)$ terms $= (2n-1)^2$ $(S_1 - S_2)^2 = \{1 + (5-3) + (9-7) +$ &c. to $(n-1)$ terms$\}^2$ $= \{1 + 2(n-1)\}^2 = (2n-1)^2$ $\therefore S_1 + S_2 = (S_1 - S_2)^2$

128. General term of $(1+x^{-2})^{-\frac{2}{3}}$

$$= (-1)^r \times \frac{p(p+q)(p+2q) \cdots \{p+(r-1)q\}}{\lfloor r \times q^r} x^r$$

$$= (-1)^r \times \frac{2\cdot 5\cdot 8 \cdots \{2+(r-1)3\}}{\lfloor r \times 3^r} (x^{-2})^r$$

$$= (-1)^r \times \frac{2\cdot 5\cdot 8 \cdots (3r-1)}{\lfloor r \times 3^r} x^{-2r}$$

When $(r+1)^{\text{th}}$ term $= 7^{\text{th}}$ term, $r = 6$

$$\therefore\ 7^{\text{th}} \text{ term} = (-1)^6 \times \frac{2\cdot 5\cdot 8\cdot 11\cdot 14\cdot 17}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6 \times 3^6} x^{-12} = \frac{2618}{6561} x^{-12}$$

When $(r-1)^{\text{th}}$ term $= 10^{\text{th}}$ term, $r = 9$

$$\therefore\ 10^{\text{th}} \text{ term} = (-1)^9 \times \frac{2\cdot 5\cdot 8\cdot 11\cdot 14\cdot 17\cdot 20\cdot 23\cdot 26}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9 \times 3^9} x^{-18} = -\frac{559130}{1594323} x^{-18}$$

129. $(x-1)(x+1)(x-2)(x+2)(x-3-\sqrt{-2})(x-3+\sqrt{-2})$ $= (x^2-1)(x^2-4)(x^2-6x+11) = x^6 - 3x^5 + 6x^4 + 30x^3 - 51x^2 - 24x + 44 = 0$

130. $(x+1)(x-1)(x-1) = x^3 - x^2 - x + 1 = 0$; then $x^5 + 2x^4 - 3x^3 - 3x^2 + 2x + 1 \div x^3 - x^2 - x + 1 = x^2 + 3x + 1$ $\therefore\ x^2 + 3x + 1 = 0$; or $x^2 + 3x = -1$ $\therefore\ x^2 + 3x + (\frac{3}{2})^2 = \frac{9}{4} - 1 = \frac{5}{4}$ $\therefore\ x + \frac{3}{2} = \pm \frac{1}{2}\sqrt{5}$, and $x = \frac{1}{2}(-3 \pm \sqrt{5})$

131. Let $x =$ the quantity, then $\dfrac{a+x}{b+x} = \dfrac{4c}{d}$ $\therefore\ ad + dx = 4bc + 4cx$; or $dx - 4cx = 4bc - ad$ $\therefore\ x = \dfrac{4bc - ad}{d - 4c}$

132. $\dfrac{2-\sqrt{3}}{2+\sqrt{3}} = \dfrac{4-2\sqrt{3}}{4+2\sqrt{3}} = \left(\dfrac{\sqrt{3}-1}{\sqrt{3}+1}\right)^2$

$$\therefore \left(\frac{2-\sqrt{3}}{2+\sqrt{3}}\right)^{\frac{1}{4}} = \left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)^{\frac{1}{2}} = \frac{(\sqrt{3}-1)^{\frac{1}{2}}(\sqrt{3}+1)^{\frac{1}{2}}}{\sqrt{3}+1} = \frac{(3-1)^{\frac{1}{2}}}{\sqrt{3}+1} = \frac{\sqrt{2}}{\sqrt{3}+1}$$

133. Multiplying the lower equation by 2 and subtracting, we have $7y - 2z = 13$ (III) $\therefore\ 3y - z + \dfrac{y}{2} = 6 + \dfrac{1}{2}$ $\therefore\ \dfrac{y-1}{2}$ is int. Let $\dfrac{y-1}{2} = t$; then $y - 1 = 2t$, and $y = 2t + 1$. Substitute this

in (ii), and $2z = 7y - 13 = 14t + 7 - 13 = 14t - 6$ $\therefore$ $z = 7t - 3$, and $y = 2t + 1$. Then, if $t = 1$, we have $y = 3$, and $z = 4$, and $x = 2$

134. $\dfrac{x^{n+1} - y^{n+1}}{x^n y^n (x - y)} = \dfrac{x^n + x^{n-1}y + x^{n-2}y^2 + x^{n-3}y^3 + \text{\&c. to } n+1 \text{ terms}}{x^n y^n}$

$= \text{when } y = x \text{ to } \dfrac{x^n + x^n + x^n + x^n + \text{\&c. to } (n+1) \text{ terms}}{x^{2n}}$

$= \dfrac{1 + 1 + 1 + \text{\&c. to } n + 1 \text{ terms}}{x^n} = \dfrac{n+1}{x^n}$

135. Let m be their *G.C.M.*, and q and q' the quotients arising from dividing them by this *G.C.M.* Then $mq + mq' = 45$ and $mqq' = 168$. Therefore $\dfrac{mq + mq'}{mqq'} = \dfrac{q + q'}{qq'} = \dfrac{45}{168} = \dfrac{15}{56}$

Whence by solving the quadratic, or by inspection, $q = 7$ and $q' = 8$, and $m = \dfrac{45}{q + q'} = \dfrac{45}{15} = 3$ $\therefore$ the numbers are $7 \times 3 = 21$, and $8 \times 3 = 24$

136. $\dfrac{1}{5} \cdot \dfrac{x^2 - 2x - 3}{x^2 - 2x - 8} + \dfrac{1}{9} \cdot \dfrac{x^2 - 2x - 15}{x^2 - 2x - 24} - \dfrac{2}{13} \cdot \dfrac{x^2 - 2x - 35}{x^2 - 2x - 48} = \dfrac{92}{585}$

Let $(x - 1)^2 = y$; then $\dfrac{1}{5} \cdot \dfrac{y - 4}{y - 9} + \dfrac{1}{9} \cdot \dfrac{y - 16}{y - 25} - \dfrac{2}{13} \cdot \dfrac{y - 36}{y - 49} = \dfrac{92}{585}$

Now $\dfrac{1}{5} + \dfrac{1}{9} - \dfrac{2}{13} = \dfrac{92}{585}$ $\therefore$ subtracting corresponding terms*

$\dfrac{1}{y - 9} + \dfrac{1}{y - 25} - \dfrac{2}{y - 49} = 0$; or $\dfrac{1}{y - 9} - \dfrac{1}{y - 49} = \dfrac{1}{y - 49} - \dfrac{1}{y - 25}$

or $\dfrac{y - 49 - y + 9}{(y - 9)(y - 49)} = \dfrac{y - 25 - y + 49}{(y - 25)(y - 49)}$; or $\dfrac{-40}{y - 9} = \dfrac{24}{y - 25}$

or $5y - 125 + 3y - 27 = 0$; or $8y = 152$ $\therefore$ $y = 19$. Then $(x + 1)^2 = 19$, $x = 1 = \pm\sqrt{19}$ and $x = 1 \pm \sqrt{19}$

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*Thus $\dfrac{1}{5} \cdot \dfrac{y - 4}{y - 9} - \dfrac{1}{5} + \dfrac{1}{9} \cdot \dfrac{y - 16}{y - 25} - \dfrac{1}{9} - \dfrac{2}{13} \cdot \dfrac{y - 36}{y - 49} + \dfrac{2}{13} = \dfrac{92}{585} - \dfrac{1}{5} - \dfrac{1}{9} + \dfrac{2}{13}$

or $\dfrac{1}{5} \dfrac{y - 4 - y + 9}{y - 9} + \dfrac{1}{9} \cdot \dfrac{y - 16 - y + 25}{y - 25} - \dfrac{2}{13} \dfrac{y - 36 - y + 49}{y - 49} = 0$

or $\dfrac{1}{5} \cdot \dfrac{5}{y - 9} + \dfrac{1}{9} \cdot \dfrac{9}{y - 25} - \dfrac{2}{13} \dfrac{13}{y - 49} = 0$, &c.

137. Let $x = v + z$, and $y = v - z$; then $2xy - 4y^2 + x^2 = 2(v^2 - z^2) - 4(v^2 - 2vz + z^2) + v^2 + 2vz + z^2 = 4$ $\therefore 10vz - v^2 - 5z^2 = 4$ Also $x^2 - y^2 = v^2 + 2vz + z^2 - v^2 + 2vz - z^2 = 4vz = 36$ $\therefore vz = 9$ $\therefore 10vz - v^2 - 5z^2 = 90 - v^2 - 5z^2 = 4$ $\therefore v^2 + 5z^2 = 86$ $\therefore v^2 \pm 2vz\sqrt{5} + 5z^2 = 86 \pm 18\sqrt{5}$. Extracting square root (right hand member by inspection, and left hand member by Art. 189) we have $v \pm z\sqrt{5} = 9 \pm \sqrt{5}$ $\therefore$ Art. 186, $v = 9$, and $\pm z\sqrt{5} = \pm\sqrt{5}$ $\therefore z = 1$ $\therefore x = 10$, $y = 8$

138. $\dfrac{1}{81} = \dfrac{1}{9^2} = 9^{-2} = (10 - 1)^{-2}$. Expanding by binomial theorem, we have $(10 - 1)^{-2} = 10^{-2} + 2 \times 10^{-3} + 3\cdot 10^{-4} +$ &c.
$+ 7\cdot 10^{\;\;1} + 8\cdot 10^{-9} + 9\cdot 10^{-10} + 10\cdot 10^{-11} +$ &c.
$+ 17\cdot 10^{-18} + 18\cdot 10^{-19} + 19\cdot 10^{-20} + 20\cdot 10^{-21} +$ &c. to infinity.
Now $8\cdot 10^{-9} + 9\cdot 10^{-10} + 10\cdot 10^{-11} + 11\cdot 10^{-12} + 12\cdot 10^{-13} +$ &c.
$= 8\cdot 10^{-9} + 9\cdot 10^{-10} + 1\cdot 10^{-10} + (1\cdot 10^{-11} + 1\cdot 10^{-12}) + (1\cdot 10^{-12} + 2\cdot 10^{-13}) + (1\cdot 10^{-13} +$ &c.)
$= 8\cdot 10^{-9} + 10\cdot 10^{-10} + 10^{-11} + 2\cdot 10^{-12} + 3\cdot 10^{-13} +$ &c.
$= 8\cdot 10^{-9} + 1\cdot 10^{-9} + 10^{-11} + 2\cdot 10^{-12} + 3\cdot 10^{-13} +$ &c.
$= 9\cdot 10^{-9} + 0\cdot 10^{-10} + 1\cdot 10^{-11} + 2\cdot 10^{-12} + 3\cdot 10^{-13} +$ &c.
Similarly for $18\cdot 10^{-19} + 19\cdot 10^{-20} + 20\cdot 10^{-21} +$ &c. and generally for $(10n - 2)10^{-10n+1}$, &c.
$\therefore (10 - 1)^{-2} = 10^{-2} + 2\cdot 10^{-3} + 3\cdot 10^{-4} +$ &c. $+ 7.10^{-8} + 9\cdot 10^{-9} + 10^{-11} + 2\cdot 10^{-12} +$ &c. $+ 7\cdot 10^{-17} + 9\cdot 10^{-18} + 10^{-20} + 2\cdot 10^{-21} +$ &c. to infinity $= \cdot 0123456790123456 79$, &c. to infinity.

NOTE.—The point in this operation is the sign of multiplication, and not the decimal point.

139. $ax^2 - bx = a^2x - ab$ $\therefore ax^2 - a^2x - bx + ab = 0$; or $(ax - b)(x - a) = 0$. Now if we assume $ax - b = 0$; or $x - a = 0$; the equation will be satisfied $\therefore ax - b = 0$; or $ax = b$ $\therefore x = \dfrac{b}{a}$ Also $x - a = 0$ $\therefore x = a$. Therefore the roots are $\dfrac{b}{a}$, and a, which are rational if a and b are rational.

140. $x^2 + (a+b)x = (n-1)ab$;

$4x^2 + 4(a+b)x + (a+b)^2 = 4(n-1)ab + (a+b)^2$

$2x + a + b = \pm\sqrt{4nab - 4ab + a^2 + 2ab + b^2} = \pm\sqrt{4nab + a^2 - 2ab + b^2}$

$2x = -a - b \pm\sqrt{4nab + (a-b)^2} \therefore x = \frac{1}{2}\{\pm\sqrt{4nab + (a-b)^2} - (a+b)\}$

141. $6x - \sqrt{x} = 1$; $x - \frac{1}{6}\sqrt{x} = \frac{1}{6}$; $x - \frac{1}{6}\sqrt{x} + \frac{1}{144} = \frac{1}{144} + \frac{24}{144} = \frac{25}{144}$

$\therefore \sqrt{x} - \frac{1}{12} = \pm\frac{5}{12} \therefore \sqrt{x} = \frac{1}{2}$ or $-\frac{1}{3} \therefore x = \frac{1}{4}$ or $\frac{1}{9}$

142. $\sqrt{x} = \sqrt{x+1} - \sqrt{x-1} \therefore x = x + 1 - 2\sqrt{x^2-1} + x - 1$

$\therefore -x = -2\sqrt{x^2-1} \therefore x^2 = 4x^2 - 4$, or $3x^2 = 4$; $x^2 = \frac{4}{3} \therefore x = \frac{2}{3}\sqrt{3}$

143. $\left.\begin{aligned} x^2 + xy + xz &= a^2 \\ xy + y^2 + yz &= b^2 \\ xz + yz + z^2 &= c^2 \end{aligned}\right\} \therefore$ by addition

$x^2 + 2xy + 2xz + y^2 + 2yz + z^2 = a^2 + b^2 + c^2$, and extracting the square root $x + y + z = \pm\sqrt{a^2+b^2+c^2}$

$$\therefore \frac{a^2}{x} = \pm\sqrt{a^2+b^2+c^2} \therefore x = \pm\frac{a^2}{\sqrt{a^2+b^2+c^2}}$$

$$\frac{b^2}{y} = \pm\sqrt{a^2+b^2+c^2} \therefore y = \pm\frac{b^2}{\sqrt{a^2+b^2+c^2}}$$

$$\frac{c^2}{z} = \pm\sqrt{a^2+b^2+c^2} \therefore z = \pm\frac{c^2}{\sqrt{a^2+b^2+c^2}}$$

144. Let $a_1 10^n + a_2 10^{n-1} + a_3 \cdot 10^{n-2} +$ &c., $+ a_{n-1}10 + a_n$ be any number $= (10^n - 1)a_1 + (10^{n-1} - 1)a_2 + (10^{n-2} - 1)a_3 +$ &c., $+ (10-1)a_{n-1} + a_1 + a_2 + a_3 \cdots + a_{n-1} + a_n$

Now $9 = 10 - 1$, and each of the coef. $(10^n - 1)$, $(10^{n-1} - 1)$, $(10^{n-2} - 1) \cdots (10 - 1)$ is divisible by $(10 - 1)$, i. e. by 9 $\therefore$ the number $= 9m + a_1 + a_2 + a_3 + \cdots\; a_{n-1} + a_n$, where m is the quotient by dividing $(10^n - 1)a_1 + (10^{n-1} - 1)a_2 + (10^{n-2} - 1)a_3 + \cdots\; (10-1)a_{n-1}$ by 9

Similarly the number reversed $= 9m^1 + a_n + a_{n-1} + a_{n-2} + \cdots\; a_3 + a_2 + a_1$.

Number $\times\, 4 = 36m + 4a_1 + 4a_2 + 4a_3 + \cdots\; 4a_{n-1} + 4a_n$

Number reversed $\times 5 = 45m^1 + 5a_n + 5a_{n-1} + 5a_{n-2} + \cdots + 5a_3 + 5a_2 + 5a_1$

$$\therefore \text{sum} = 36m + 45m^1 + 9a_1 + 9a_2 + 9a_3 + \cdots + 9a_{n-1} + 9a_n$$
$$= 9\{4m + 5m^1 + a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n\}$$

This statement may be generalized as follows:—

General Theorem.—*Let* r *be the radix of any system of numbers, then if any number in that system be multiplied by any number* n *and the same number reversed, as to its orders, be multiplied by* r − (n + 1); *then the sum of the two products thus obtained is divisible by* (r − 1).

145. $(a + b)(b + c) - (a + 1)(c + 1) - (a + c)(b - 1)$
$= ab + ac + bc + b^2 - ac - a - c - 1 - ab + a - bc + c = b^2 - 1$

146. $\left\{\left(\frac{xy}{3}\right)^2 - 3\left(\frac{xy}{3}\right) + 3^2\right\}\left(\frac{xy}{3} + 3\right) = \left(\frac{xy}{3}\right)^3 + 3^3 = \frac{x^3y^3}{27} + 27$

(147)

$$\frac{a^2x^2+2abxy+b^2y^2+c^2x^2+2cdxy+d^2y^2+a^2y^2-2abxy+b^2x^2+c^2y^2-2cdxy+d^2x^2}{x^2+y^2}$$
$$= \frac{a^2(x^2 + y^2) + b^2(x^2 + y^2) + c^2(x^2 + y^2) + d^2(x^2 + y^2)}{x^2 + y^2}$$
$$= \frac{(a^2 + b^2 + c^2 + d^2)(x^2 + y^2)}{x^2 + y^2} = a^2 + b^2 + c^2 + d^2$$

148. $\sqrt{\{a^2(x^2 + 4x + 4) - 2a(x + 2) + 1\}} = \pm\{a(x + 2) - 1\}$

149. *G.C.M.* of $a^2 + 2ab + b^2 - c^2$, and $a^2 - b^2 + 2bc - c^2$; that is of $(a + b)^2 - c^2$, and $a^2 - (b - c)^2$; that is of $(a + b - c)(a + b + c)$, and $(a - b + c)(a + b - c)$ is evidently $a + b - c$

150.
$$\begin{array}{r|r|l}
1 & 4 + 0 & + 5 + 0 + 1 \\
+ 0 & 0 & + 0 \\
- 2 & & - 8 + 0 \\
+ 1 & & \quad + 4 + 0 \\
\hline
 & 4 + 0 & - 3 + 4 + 1 = 4x - \dfrac{3x^2 - 4x - 1}{x^3 + 2x - 1}
\end{array}$$

$$
\begin{array}{r|l}
1 & 4 + 0 + 5 + 0 + 1 \\
+0 & \quad + 0 + 0 + 0 + 0 + 0 \\
-2 & \qquad - 8 + 0 + 6 - 8 - 14 \\
+1 & \qquad\quad + 4 + 0 - 3 + 4 + 7 \\
\hline
 & 4 + 0 - 3 + 4 + 7 - 11 - 10
\end{array}
$$

$$4x - 3x^{-1} + 4x^{-2} + 7x^{-3} - 11x^{-4} - 10x^{-5} + \&c.$$

151.
$$
\begin{array}{r|l}
1 & 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 \\
+1 & \quad + 1 + 1 + 0 - 1 - 1 + 0 + 1 \\
-1 & \qquad - 1 - 1 + 0 + 1 + 1 + 0 \\
\hline
 & 1 + 1 + 0 - 1 - 1 + 0 + 1 + 1 + \&c.
\end{array}
$$

$$= 1 + x - x^3 - x^4 + x^6 + x^7 - x^9 - x^{10} + \&c.$$

152. $$\left(\frac{a}{a+b} + \frac{b}{a-b}\right) \times \left(\frac{a}{a-b} - \frac{b}{a+b}\right) = \frac{a^2+b^2}{(a+b)(a-b)} \times \frac{a^2+b^2}{(a-b)(a+b)}$$

$$= \frac{(a^2+b^2)^2}{(a^2+b^2)^2} = \frac{a^4 + 2a^2b^2 + b^4}{a^4 - 2a^2b^2 + b^4}$$

153. $$\frac{c(a-b)}{(a+c)(b+c)} \div \frac{c(a-b)}{(a+c)(b+c)} = \frac{c(a-b)}{(a+c)(b+c)} \times \frac{(a+c)(b+c)}{c(a-b)} = 1$$

154. $$\frac{3(x-2)}{(x-1)(x-3)} - \frac{x-3}{(x-1)(x-3)} - \frac{x-1}{(x-1)(x-3)} - \frac{1}{x-2}$$

$$= \frac{3(x-2) - x + 3 - x + 1}{(x-1)(x-3)} - \frac{1}{x-2} = \frac{x-2}{(x-1)(x-3)} - \frac{1}{x-2}$$

$$= \frac{(x-2)^2 - (x-1)(x-3)}{(x-1)(x-3)(x-2)} = \frac{x^2 - 4x + 4 - x^2 + 4x - 3}{(x-1)(x-2)(x-3)}$$

$$= \frac{1}{(x-1)(x-2)(x-3)}$$

155. $$\frac{\{(xy+1) + 2x\}\{(xy+1) + 2y\} + (x-y)^2}{x^2y^2 + 1 - x^2 - y^2}$$

$$= \frac{x^2y^2 + 2xy + 1 + 2x(xy+1) + 2y(xy+1) + (x+y)^2}{x^2y^2 + 1 - x^2 - y^2}$$

$$= \frac{(xy+1)^2 + 2(xy+1)(x+y) + (x+y)^2}{(x^2y^2 + 2xy + 1) - (x^2 + 2xy + y^2)} = \frac{(xy+1+x+y)^2}{(xy+1)^2 - (x+y)^2}$$

$$= \frac{(xy+x+y+1)^2}{(xy+x+y+1)(xy-x-y+1)} = \frac{xy+x+y+1}{xy-x-y+1} = \frac{(x+1)(y+1)}{(x-1)(y-1)}$$

156. (I) $\left.\begin{array}{l} ax^2 + bx + c = 0 \\ a_1x^2 + b_1x + c_1 = 0 \end{array}\right\}$ Divide by coefficients of x^2; then

$\left.\begin{array}{l} x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = 0 \\ x^2 + \dfrac{b_1}{a_1} \quad + \dfrac{c_1}{a_1} = 0 \end{array}\right\}$ Let r and r_1, be roots of first, and r and r_2, the roots of 2nd equat.;

Then $x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = (x - r)(x - r_1) = 0$

And $x^2 + \dfrac{b_1}{a_1}x + \dfrac{c_1}{a} = (x - r)(x - r_2) = 0$

Hence in order that the equations may have a common root, they must have a common measure.

(II) Having divided as before by coef. of x^2, let r and r_1 = roots of one, and $-r$ and $-r_1$ the roots of the other equation.

Then $x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = (x - r)(x - r_1) = x^2 - (r + r_1)x + rr_1 = 0$

And $x^2 + \dfrac{b_1}{a_1} \quad + \dfrac{c_1}{a_1} = (x + r)(x + r_1) = x^2 + (r + r_1)x + rr_1 = 0$

$\therefore \dfrac{b}{a} = -(r + r_1) = -\dfrac{b_1}{a_1}$, and $\dfrac{c}{a} = rr_1 = \dfrac{c_1}{a_1}$. Hence in order that the roots may be equal in magnitude but opposite in signs, the coefficients of x must be equal in magnitude but opposite in sign, and the coefficients of x^2 and also of x^0 must be equal.

157. $\dfrac{2(x + 1) - (3x + 4)}{4} = \dfrac{(2x - 1) - (5x - 6)}{3} \therefore -3x - 6 = -12x + 20 \therefore x = 2\frac{8}{9}$

158. $(x - 1)^2(x + 4) = (x + 3)^2(x - 2) \therefore x^3 + 2x^2 - 7x + 4 = x^3 + 4x^2 - 3x - 18 \therefore x^2 + 2x = 11$; $\sqrt{x^2 + 2x + 1} = \pm\sqrt{12} = \pm 2\sqrt{3}$ $\therefore x = 1 \pm 2\sqrt{3}$

159. $\dfrac{1 + 2x}{1 - 2x} = \dfrac{2 + 2x + 2\sqrt{1 + 2x}}{2 - 2x + 2\sqrt{1 - 2x}}$; extracting the square root of each side, $\dfrac{\sqrt{1 + 2x}}{\sqrt{1 - 2x}} = \pm\dfrac{\sqrt{1 + 2x} + 1}{\sqrt{1 - 2x} - 1}$; clearing of fractions,

$\sqrt{1-4x^2} - \sqrt{1+2x} = \pm(\sqrt{1-4x^2} + \sqrt{1-2x})$

$\therefore$ (I) $\sqrt{1-4x^2} - \sqrt{1+2x} = \sqrt{1-4x^2} + \sqrt{1-2x}$; or $-\sqrt{1+2x} = \sqrt{1-2x}$ $\therefore 1+2x = 1-2x$ $\therefore 2x = -2x$; or $x = 0$

And (II) $\sqrt{1-4x^2} - \sqrt{1+2x} = -\sqrt{1-4x^2} - \sqrt{1-2x}$

$\therefore 2\sqrt{1-4x^2} = \sqrt{1+2x} - \sqrt{1-2x}$

Squaring, $4(1-4x^2) = 2 - 2\sqrt{1-4x^2}$ $\therefore 2(1-4x^2) + (1-4x^2)^{\frac{1}{2}} = 1$

$\therefore (1-4x^2) + \frac{1}{2}(1-4x^2)^{\frac{1}{2}} = \frac{1}{2}$ $\therefore (1-4x^2) + \frac{1}{2}(1-4x^2)^{\frac{1}{2}} + \frac{1}{16} = \frac{9}{16}$

$\therefore (1-4x^2)^{\frac{1}{2}} + \frac{1}{4} = \pm\frac{3}{4}$ $\therefore (1-4x^2)^{\frac{1}{2}} = \frac{1}{2}$ or -1 $\therefore 1-4x^2 = \frac{1}{4}$ or 1

$\therefore 4x^2 = \frac{3}{4}$ or 0; $2x = \pm\frac{1}{2}\sqrt{3}$ or 0 $\therefore x = \pm\frac{1}{4}\sqrt{3}$ or 0

160. $\dfrac{(n-1)^2x^2 - 2(n-1)^2x + (n-1)^2 + 4n}{(n-1)^2x^2 + 2(n-1)^2x + (n-1)^2 + 4n} = P$

$\therefore \dfrac{(n-1)^2x^2 - 2(n-1)^2x + (n+1)^2}{(n-1)^2x + 2(n-1)^2x + (n+1)^2} = \dfrac{P}{1}$ $\therefore$ Art. 105 (VII)

$\dfrac{2(n-1)^2x^2 + 2(n+1)^2}{-4(n-1)^2x} = \dfrac{P+1}{P-1}$; or $\dfrac{(n-1)^2x^2 + (n+1)^2}{-2(n-1)^2x} = \dfrac{P+1}{P-1}$

$\therefore (n-1)^2(P-1)x^2 + (n+1)^2(P-1) = -2(n-1)^2(P+1)x$;

or $(n-1)^2(1-P)x^2 - 2(n-1)^2(1+P)x = -(n+1)^2(1-P)$;

$4(n-1)^4(1-P)^2x^2 - 8(n-1)^4(1^2-P^2)x + 4(n-1)^4(1+P)^2$

$= 4(n-1)^4(1+P)^2 - 4(n-1)^2(n+1)^2(1-P)^2$. Dividing by $4(n-1)^2$, $(n-1)^2(1-P^2)x^2 - 2(n-1)^2(1-P^2)x + (n-1)^2(1+P)^2$

$= (n-1)^2(1+P)^2 - (n+1)^2(1-P)^2$

$\therefore (n-1)(1-P)x - (n-1)(1+P) = \pm\sqrt{(n-1)^2(1+P)^2 - (n+1)^2(1-P)^2}$

$\therefore (n-1)(1-P)x = (n-1)(1+P) \pm \sqrt{4Pn^2 - 4P^2n - 4n + 4P}$

$\therefore (n-1)(1-P)x = (n-1)(1+P) \pm 2\sqrt{(Pn-1)(n-P)}$

$\therefore x = \dfrac{1+P}{1-P} \pm \dfrac{2}{(n-1)(1-P)}\sqrt{(Pn-1)(n-P)}$; where in order that x may be real $\sqrt{(Pn-1)(n-P)}$ must be real, that is $(Pn-1)(n-P)$ must be positive, and if n is positive, in order that $(Pn-1)(n-P)$ may be positive, P must neither be $> n$ nor $< \dfrac{1}{n}$

161. $A = \frac{1}{2}(a+b) = \frac{1}{2}(\frac{3}{4}+\frac{4}{3}) = \frac{1}{2} \times \frac{25}{12} = \frac{25}{24} = 1\frac{1}{24}$

$$H = \frac{2ab}{a+b} = \frac{2 \times \frac{3}{4} \times \frac{4}{3}}{\frac{3}{4}+\frac{4}{3}} = \frac{2}{\frac{25}{12}} = \frac{24}{25}$$

$$G = \sqrt{ab} = \sqrt{\frac{3}{4} \times \frac{4}{3}} = \sqrt{1} = 1$$

162. $H = \dfrac{2ab}{a+b}$ $\therefore$ $Ha + Hb = 2ab$ $\therefore$ $Ha - ab = ab - Hb$ or $(H-b)a = (a-H)b$ $\therefore \dfrac{a}{a-H} = \dfrac{b}{H-b}$ $\therefore \dfrac{H-H+a}{a-H} = \dfrac{H-H+b}{H-b}$

$$\therefore \frac{H-H-a}{H-a} = \frac{H-H+b}{H-b} \quad \therefore \frac{(H-a)-H}{H(H-a)} = \frac{H-(H-b)}{H(H-b)}$$

$$\therefore \frac{H-a}{H(H-a)} - \frac{H}{H(H-a)} = \frac{H}{H(H-b)} - \frac{(H-b)}{H(H-b)};$$

that is $\dfrac{1}{H} - \dfrac{1}{H-a} = \dfrac{1}{H-b} - \dfrac{1}{H}$ $\therefore \dfrac{1}{H-a}, \dfrac{1}{H}$ and $\dfrac{1}{H-b}$ are in A. prog. And $\therefore$ $H-a$, H and $H-b$ are in H. progression; that is, H is the H mean between $H-a$ and $H-b$

163. The n^{th} term $= a + (n-1)d$ $\therefore$ 37^{th} term of the series $\frac{36}{6} + \frac{35}{6} + \frac{34}{6} +$ &c. $= \frac{36}{6} + (37-1)(-\frac{1}{6}) = \frac{36}{6} - \frac{36}{6} = 0$

$S_{31} = \{2 \times 6 + (31-1)(-\frac{1}{6})\}\frac{31}{2} = (12 + 30 \times -\frac{1}{6})\frac{31}{2} = 7 \times \frac{31}{2} = 108\frac{1}{2}$

$S_{42} = \{2 \times 6 + (42-1)(-\frac{1}{6})\}\frac{42}{2} = (12 + 41 \times -\frac{1}{6})21 = (12 - 6\frac{5}{6})21$

$= 5\frac{1}{6} \times 21 = 108\frac{1}{2}$ $\therefore$ $S_{31} + S_{42} = 108\frac{1}{2} + 108\frac{1}{2} = 217$

164. $S_n = \dfrac{a(1-r^n)}{1-r} = \dfrac{3\frac{1}{3}\{1-(\frac{3}{5})^n\}}{1-\frac{3}{5}} = \dfrac{\frac{10}{3}}{\frac{2}{5}}\{1-(\frac{3}{5})^n\} = \frac{25}{3}\{1-(\frac{3}{5})^n\}$ $= \dfrac{25}{3} - \dfrac{3^{n-1}}{5^{n-2}}$

165. $S_\infty = \dfrac{a}{1-r} = \dfrac{1}{1(-\frac{4}{10})} = \dfrac{1}{1+\frac{2}{5}} = \dfrac{5}{7}$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{1-(-\cdot 4)^n}{1-(-\cdot 4)} = \frac{1-(-\cdot 4)^n}{1+\frac{2}{5}} = \frac{5}{7}\left\{1 - \left(-\frac{2}{5}\right)^n\right\}$$

$S_\infty - S_n = \frac{5}{7} - \frac{5}{7}\{1-(-\frac{2}{5})^n\} = \frac{5}{7} - \frac{5}{7} + \frac{5}{7}(-\frac{2}{5})^n = \frac{5}{7}(-\frac{2}{5})^n$

166. Of 1^{st} series, $S_n = \{2+(n-1)1\}\dfrac{n}{2} = (n+1)\dfrac{n}{2}$, and $S_p = (p+1)\dfrac{p}{2}$

Of 2^{nd} series $S_n = \{4+(n-1)3\}\dfrac{n}{2} = (3n+1)\dfrac{n}{2}$, and $S_p = (3p+1)\dfrac{p}{2}$

Of 3^{rd} series $S_n = \{6 + (n-1)5\}\frac{n}{2} = (5n+1)\frac{n}{2}$, and $S_p = (5p+1)\frac{p}{2}$

Of 4^{th} series $S_n = \{8 + (n-1)7\}\frac{n}{2} = (7n+1)\frac{n}{2}$, and $S_p = (7p+1)\frac{p}{2}$

$\therefore$ of the series $(n+1)\frac{n}{2} + (3n+1)\frac{n}{2} + (5n+1)\frac{n}{2} + \&c.$, where the first term is $(n+1)\frac{n}{2}$ and the common difference is $2n \times \frac{n}{2} = n^2$, the $S_p = \{2(n+1)\frac{n}{2} + (p-1)n^2\}\frac{p}{2} = (n^2 + n + pn^2 - n^2)\frac{p}{2}$ $= (n + pn^2)\frac{p}{2} = (1 + pn)\frac{pn}{2}$

Also of the series $(p+1)\frac{p}{2} + (3p+1)\frac{p}{2} + (5p+1)\frac{p}{2} + \&c.$ where the 1^{st} term is $(p+1)\frac{p}{2}$ and the common difference is $2p \times \frac{p}{2} = p^2$ $S_n = \{2(p+1)\frac{p}{2} + (n-1)p^2\}\frac{n}{2} = (p^2 + p + np^2 - p^2)\frac{n}{2} = (p + np^2)\frac{n}{2}$ $= (1 + pn)\frac{pn}{2}$ $\therefore$ S_p of the former series $= S_n$ of the latter series.

167. $\{(x+y) - \sqrt{xy}\}\{(x+y) + \sqrt{xy}\} = (x^2 + 2xy + y^2) - xy$ $= x^2 + xy + y^2$

$\{(x^2+y^2) + xy\}\{(x^2+y^2) - xy\} = (x^2+y^2)^2 - x^2y^2 = x^4 + x^2y^2 + y^4$

168. $8(5^2 - 3 \times 8)^{\frac{1}{2}} + 5(5^2 + 3 \times 8)^{\frac{1}{2}} = 8(25 - 24)^{\frac{1}{2}} + 5(25 + 24)^{\frac{1}{2}}$ $= 8\sqrt{1} + 5\sqrt{49} = 8 + 35 = 43$

169. $\sqrt{16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4} = 4a^2 - 12ab + 9b^2$, and $\sqrt{4a^2 - 12ab + 9b^2} = 2a - 3b$

170. $\frac{1}{a} + \frac{1}{4d} = \frac{1}{ad}\left(\frac{a}{4} + d\right)$. Also $-\frac{1}{2b} - \frac{1}{3c} = -\frac{3c}{6bc} - \frac{2b}{6bc}$; but since $\frac{a}{b} = \frac{c}{d}$, it follows that $bc = ad$ $\therefore$ $-\frac{3c}{6bc} - \frac{2b}{6bc}$ $= -\frac{3c}{6ad} - \frac{2b}{6ad} = -\frac{c}{2ad} - \frac{b}{3ad} = -\left(\frac{c}{2} \times \frac{1}{ad}\right) - \left(\frac{b}{3} \times \frac{1}{ad}\right)$

$\therefore \frac{1}{a} - \frac{1}{2b} - \frac{1}{3c} + \frac{1}{4d} = \frac{1}{ad}\left(\frac{a}{4} + d\right) - \frac{c}{2} \times \frac{1}{ad} - \frac{b}{3} \times \frac{1}{ad}$ $= \frac{1}{ad}\left(\frac{a}{4} - \frac{b}{3} - \frac{c}{2} + d\right)$

171. Multiply by $4(x+1)$, and $8x + 12 = 4x + 5 + \dfrac{12(x+1)^2}{3x+1}$

Reducing and then clearing of fractions, we have $12x^2 + 25x + 7 = 12x^2 + 24x + 12 \therefore x = 5$

172. $\dfrac{2x+b}{x^2+bx} = \dfrac{2a+b}{a^2+ab} \therefore$ clearing of fractions,

$\therefore 2a^2x + 2abx + a^2b + ab^2 = 2ax^2 + 2abx + bx^2 + b^2x$

$\therefore 2ax^2 + bx^2 - 2a^2x + b^2x = a^2b + ab^2$

$(2a+b)x^2 - (2a^2 - b^2)x = ab(a+b)$

$4(2a+b)^2x^2 - 4(2a+b)(2a^2-b^2) + (2a^2-b^2)^2$

$= 4ab(a+b)(2a+b) + (2a^2-b^2)^2 = 4a^4 + 8a^3b + 8a^2b^2 + 4ab^3 + b^4$

$\therefore 2(2a+b)x - (2a^2 - b^2) = \pm\sqrt{4a^4 + 8a^3b + 8a^2b^2 + 4ab^3 + b^4}$

$= \pm(2a^2 + 2ab + b^2) \therefore 2(2a+b)x = 2a^2 - b^2 \pm (2a^2 + 2ab + b^2)$

$= 4a^2 + 2ab$, or $= -2ab - 2b^2 = 2a(2a+b)$, or $= -2b(a+b)$

$\therefore x = \dfrac{2a(2a+b)}{2(2a+b)} = a$, or $x = -\dfrac{b(a+b)}{2a+b}$

173. Since $x = \dfrac{a+1}{ab+1}$, and $y = \dfrac{ab+a}{ab+1}$; $x + y = \dfrac{a+1}{ab+1} + \dfrac{ab+a}{ab+1}$

$= \dfrac{2a+ab+1}{ab+1}$

$$\therefore \frac{x+y-1}{x+y+1} = \frac{\dfrac{2a+ab+1}{ab+1} - 1}{\dfrac{2a+ab+1}{ab+1} + 1} = \frac{\dfrac{2a+ab+1-ab-1}{ab+1}}{\dfrac{2a+ab+1+ab+1}{ab+1}}$$

$$= \frac{2a}{2a+2ab+2} = \frac{a}{a+ab+1}$$

174. $2x^2 - 2xz - 2xy + 2yz + 2y^2 - 2xy - 2yz + 2xz + 2z^2 - 2xz + 2xy - 2yz = 2x^2 + 2y^2 + 2z^2 - 2xy - 2xz - 2yz$

$= (x^2 - 2xy + y^2) + (x^2 - 2xz + z^2) + (y^2 - 2yz + z^2)$

$= (x-y)^2 + (x-z)^2 + (y-z)^2$

175. $(a+b)^2 - c^2 \,)\, (a^2-b^2)^2 + 4abc^2 - c^4 \,(\, (a-b)^2 + c^2$

$(a^2-b^2)^2 - (a-b)^2c^2$

$(a+b)^2c^2 - c^4$

$(a+b)^2c^2 - c^4$

176. $x^8 - 1\big)\, x^{10} + x^9 + x^8 + 2x^7 + 2x^4 + 2x^3 + x^2 + x + 1 \big(\, x^2 + x + 1$

$$
\begin{array}{l}
x^{10} - x^2 \\
\hline
x^9 + 2x^2 \\
x^9 - x \\
\hline
x^8 + 2x^2 + 2x \\
x^8 - 1 \\
\hline
2x^7 + 2x^4 + 2x^3 + 2x^2 + 2x + 2
\end{array}
$$

$x^7 + x^4 + x^3 + x^2 + x + 1\big)\, x^8 - 1 \big(\, x$

$$
\begin{array}{l}
x^8 + x^5 + x^4 + x^3 + x^2 + x \\
\hline
- x^5 - x^4 - x^3 - x^2 - x - 1
\end{array}
$$

$x^5 + x^4 + x^3 + x^2 + x + 1\big)\, x^7 + x^4 + x^3 + x^2 + x + 1 \big(\, x^2 - x$

$$
\begin{array}{l}
x^7 + x^6 + x^5 + x^4 + x^3 + x^2 \\
\hline
- x^6 - x^5 + x + 1 \\
- x^6 - x^5 - x^4 - x^3 - x^2 - x \\
\hline
x^4 + x^3 + x^2 + 2x + 1
\end{array}
$$

$x^4 + x^3 + x^2 + 2x + 1\big)\, x^5 + x^4 + x^3 + x^2 + x + 1 \big(\, x$

$$
\begin{array}{l}
x^5 + x^4 + x^3 + 2x^2 + x \\
\hline
- \quad x^2 + 1
\end{array}
$$

$x^2 - 1\big)\, x^4 + x^3 + x^2 + 2x + 1 \big(\, x^2 + x + 2$

$$
\begin{array}{l}
x^4 - x^2 \\
\hline
x^3 + 2x^2 \\
x^3 - x \\
\hline
2x^2 + 3x \\
2x^2 - 2 \\
\hline
3x + 3 \\
3(x + 1)
\end{array}
$$

G.C.M. $= x + 1\big)\, x^2 - 1 \big(\, x - 1$

$$
\begin{array}{l}
x^2 + x \\
\hline
- x - 1 \\
- x - 1 \\
\hline
\end{array}
$$

177. $\dfrac{(2x + 3)(x - 1)}{(x + 5)(x^2 - 1)} - \dfrac{(x - 7)(x + 1)}{(x + 5)(x^2 - 1)} - \dfrac{x + 2}{x^2 + 1}$

$$= \frac{2x^2 + x - 3 - (x^2 - 6x - 7)}{(x + 5)(x^2 - 1)} - \frac{x + 2}{x^2 + 1} = \frac{x^2 - 7x + 4}{(x + 5)(x^2 - 1)} - \frac{x + 2}{x^2 + 1}$$

$$= \frac{(x^2 + 7x + 4)(x^2 + 1) - (x + 2)(x^2 - 1)(x + 5)}{(x + 5)(x^4 - 1)}$$

$$= \frac{(x^2 + 7x + 4)x^2 + x^2 + 7x + 4 - (x^2 + 7x + 10)x^2 + x^2 + 7x + 10}{(x + 5)(x^4 - 1)}$$

$$= \frac{2x^2 + 14x + 14 - 6x^2}{(x + 5)(x^4 - 1)} = \frac{14x - 4x^2 + 14}{(x + 5)(x^4 - 1)}$$

(178)

$$\frac{(a+b\sqrt{-1})^2 + (a-b\sqrt{-1})^2}{(a-b\sqrt{-1})(a+b\sqrt{-1})} = \frac{a^2+2ab\sqrt{-1}-b^2+a^2-2ab\sqrt{-1}-b^2}{a^2-b^2(\sqrt{-1})^2}$$

$$= \frac{2a^2-2b^2}{a^2+b^2} = \frac{2(a^2-b^2)}{a^2+b^2}$$

179. Let $x-3y$ be the first of any four positive quantities in *A.P.*, and let $2y$ be their common difference.
Then the four quantities are $x-3y$, $x-y$, $x+y$ and $x+3y$. And the sum of the extremes $= x-3y+x+3y = 2x$. Also the sum of the means $= x-y+x+y = 2x$. And $2x = 2x$
$\therefore$ the sum of the extremes = the sum of the means.

Again let $\frac{x}{y^2}$ be the first of four positive quantities in *G.P.*, and let y by their common ratio.

Then the four quantities are $\frac{x}{y^2}$, $\frac{x}{y}$, x and xy.

Sum of extremes $= \frac{x}{y^2} + xy = \frac{x+xy^3}{y^2}$; sum of means $= \frac{x}{y} + x = \frac{x+xy}{y}$

Then $\frac{x+xy^3}{y^2} \gtreqless \frac{x+xy}{y}$, according as $x+xy^3 \gtreqless xy+xy^2$;

or as $1+y^3 \gtreqless y+y^2$; or as $(1+y)(1-y+y^2) \gtreqless y(1+y)$;

or as $1-y+y^2 \gtreqless y$; or as $1+y^2 \gtreqless 2y$. But $1+y^2 > 2y$ by Art. 134 Note 2, $\therefore \frac{x+xy^3}{y^2} > \frac{x+xy}{y}$, that is the sum of the extremes is greater than the sum of the means.

Lastly, if as before $x-3y$, $x-y$, $x+y$ and $x+3y$ are in *A.P.*, their reciprocals $\frac{1}{x-3y}$, $\frac{1}{x-y}$, $\frac{1}{x+y}$ and $\frac{1}{x+3y}$ are in *H.P.*

Then $\frac{1}{x-3y} + \frac{1}{x+3y} = \frac{2x}{x^2-9y^2}$ = sum of extremes.

And $\frac{1}{x-y} + \frac{1}{x+y} = \frac{2x}{x^2-y^2}$ = sum of means.

Now whether y be positive or negative, y^2 is necessarily positive, and therefore $x^2 - y^2 > x^2 - 9y^2$, and $\therefore \dfrac{2x}{x^2 - 9y^2} > \dfrac{2x}{x^2 - y^2}$; that is the sum of the extremes is greater than the sum of the means.

180. $S_{2n-1} = \{2a + (2n-2)d\}\dfrac{2n-1}{2}$, and when $d = a$ $S_{2n-1} = \{2a - (2n-2)a\}\dfrac{2n-1}{2} = \{2a + 2an - 2a\}\dfrac{2n-1}{2} = na(2n-1)$ Also $(2n-1)^{th}$ term $= a + (2n-2)d = a + (2n-2)a = a + 2an - 2a$ $= a(2n-1)$ $\therefore$ sum of $2n-1$ terms $=$ the $(2n-1)^{th}$ term $\times n$ when the series is ascending, i. e. when the first term is the least and the last term is the greatest.

181. $ab + b\sqrt{a^2 - x^2} = x^2$ $\therefore$ $b\sqrt{a^2 - x^2} = x^2 - ab$; $a^2b^2 - b^2x^2$ $= x^4 - 2abx^2 + a^2b^2$ $\therefore$ $x^4 - 2abx^2 + b^2x^2 = 0$; or $x^2(x^2 - 2ab + b^2) = 0$ $\therefore$ $x = 0$ or $x = \pm\sqrt{b(2a-b)}$

182. $3x^{\frac{5}{3}} + x^{\frac{5}{6}} = 3104$; $36x^{\frac{5}{3}} + 12x^{\frac{5}{6}} + 1 = 37249$; $6x^{\frac{5}{6}} + 1 = \pm 193$; $6x^{\frac{5}{6}} = 192$ or -194; $x^{\frac{5}{6}} = 32$ or $-32\frac{1}{3}$; $x^{\frac{1}{6}} = 2$, hence $x = 64$; or $x^5 = (-32\frac{1}{3})^6$, whence $x = \sqrt[5]{(32\frac{1}{3})^6} = 32\frac{1}{3}\sqrt[5]{32\frac{1}{3}} = \frac{97}{3}\sqrt[5]{\frac{97}{3}}$ $= \frac{97}{3}\sqrt[5]{\frac{7857}{243}} = \frac{97}{9}\sqrt[5]{7857}$

183. $\dfrac{x^2 + 2ax + x^2 - x^2 + 2ax - a^2}{x^2 - a^2} = \dfrac{b^2 + 2bx + x^2 - b^2 + 2bx - x^2}{b^2 - x^2}$

$\therefore \dfrac{4ax}{x^2 - a^2} = \dfrac{4bx}{b^2 - x^2}$ $\therefore \dfrac{a}{x^2 - a^2} = \dfrac{b}{b^2 - x^2}$; $ab^2 - ax^2 = bx^2 - ba^2$, or $bx^2 + ax^2 = ab^2 + ba^2$ $\therefore$ $(b + a)x^2 = ab(b + a)$ $\therefore$ $x^2 = ab$, whence $x = \pm\sqrt{ab}$

184. $\sqrt{x^2 + \sqrt{x^2 + 96}} = 11 - x$ $\therefore$ $x^2 + \sqrt{x^2 + 96} = 121 - 22x + x^2$ $\therefore$ $\sqrt{x^2 + 96} = 121 - 22x$. Again squaring

$x^2 + 96 = 14641 - 5324x + 484x^2$ $\therefore$ $483x^2 - 5324x = -14545$;

or $x^2 - \dfrac{5324}{483}x = -\dfrac{14545}{483}$

$\therefore x^2 - \dfrac{5324}{483}x + \left(\dfrac{2662}{483}\right)^2 = \dfrac{7086244}{233289} - \dfrac{14545}{483} = \dfrac{7086244 - 7025235}{233289}$

$$\therefore\ x - \frac{2662}{483} = \pm\sqrt{\frac{61009}{233289}} = \pm\frac{247}{483}$$

$$\therefore\ x = \frac{2662 \pm 247}{483} = \frac{2909}{483};\ \text{or}\ \frac{2415}{483} = 6\tfrac{11}{483}\ \text{or}\ 5$$

185. Let x = the left hand digit, and y = the right hand digit; then the number is $10x + y$

$\therefore\ \dfrac{10x+y}{x-y} = 21$, whence $10x + y = 21x - 21y$, or $22y = 11x$, or $x = 2y$

Also $\dfrac{10x+y}{x+y} + 17 = 10y + x$, whence $27x + 18y = 11xy + 10y^2 + x^2$

But $x = 2y$; substituting this in the last equation, we have $54y + 18y = 22y^2 + 10y^2 + 4y^2$ $\therefore$ $72 = 36y$, whence $y = 2$

And $x = 2y = 4$ $\therefore$ the required number is 42

186. Let x = minutes per mile taken by B, then $x + 1$ = minutes per mile taken by A; $\dfrac{60}{x}$ = miles per hour of B, and $\dfrac{60}{x+1}$ = miles per hour taken by A.

The second time round the rate per hour of $B = \dfrac{60}{x} - 2 = \dfrac{60 - 2x}{x}$, and rate per hour of $A = \dfrac{60}{x+1} + 2 = \dfrac{62 + 2x}{x+1}$

And since the course is 2 miles long, the time in hours taken by B to go round $= \dfrac{2}{\frac{62+2x}{x+1}} = \dfrac{1}{\frac{31+x}{x+1}} = \dfrac{x+1}{31+x}$ $\therefore$ time in minutes required to go round $= \dfrac{60x+60}{31+x}$ $\therefore$ minutes per mile taken by $A = \dfrac{30x+30}{31+x}$; similarly minutes per mile in 2nd round required by $B = \dfrac{30x}{30-x}$, and since A does the two miles in two minutes less than B, his time per mile will be one minute less than B $\therefore$ $\dfrac{30x+30}{31+x} + 1 = \dfrac{30x}{30-x}$, whence by reduction

$\frac{30x + 30 + 31 + x}{31 + x} = \frac{31x+61}{31+x} = \frac{30x}{30 - x} \therefore 930x + 1830 - 31x^2 - 61x$
$= 930x + 30x^2$; or $-61x^2 - 61x = -1830$, whence $x^2 + x = 30$;
$x^2 + x + \frac{1}{4} = \frac{121}{4} \therefore x + \frac{1}{2} = \pm \frac{11}{2}$, and $x = 5$

$\therefore$ A's rate 1st round $= 5 + 1 = 6$ min. per mile $= 10$ miles per hour

B's rate 1st round $= 5$ minutes per mile, or 12 miles per hour

A's rate 2nd time round $= 10 + 2 = 12$ miles per hour

B's rate 2nd time round $= 12 - 2 = 10$ miles per hour

Whole time of B for both rounds $= 10 + 12 = 22$ minutes
Whole time of A for both rounds $= 12 + 10 = 22$ minutes
$\therefore$ neither horse wins.

187. Let x, $x + 1$, $x + 2$, $x + 3$ and $x + 4$ be any five consecutive integers; then $x(x + 2)(x + 4) + (x + 1)^3 + (x + 3)^3$
$= (x + 2)(x^2 + 4x) + x^3 + 3x^2 + 3x + 1 + x^3 + 9x^2 + 27x + 27$
$= (x+2)(x^2+4x) + (x^3+4x^2+5x+2) + (4x+8) + (x^3+8x^2+21x+18)$
$= (x+2)(x^2+4x) + (x+2)(x^2+2x+1) + (x+2)4 + (x+2)(x^2+6x+9)$
$= (x + 2)\{(x^2 + 4x) + (x^2 + 2x + 1) + 4 + (x^2 + 6x + 9)\}$
$= (x + 2)\{(x^2 + 2x + 1) + (x^2 + 4x + 4) + (x^2 + 6x + 9)\}$
$= (x + 2)\{(x + 1)^2 + (x + 2)^2 + (x + 3)^2\}$ = product of middle number by the sum of the squares of the middle three.

188. $x^4 + y^4 + x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
$= 2x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 2y^4 = 2(x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4)$
$= 2(x^2 + xy + y^2)^2$

189. $(x^3 + y^3 + x^2y + xy^2)(x^3 - y^3 - x^2y + xy^2)$
$= \{(x^3 + xy^2) + (y^3 + x^2y)\}\{(x^3 + xy^2) - (y^3 + x^2y)\}$
$= (x^3 + xy^2)^2 - (y^3 + x^2y)^2 = x^6 + 2x^4y^2 + x^2y^4 - (y^6 + 2x^2y^4 + x^4y^2)$
$= x^6 + x^4y^2 - x^2y^4 - y^6$

190. $x^2 = (\sqrt{a+b} \pm \sqrt{a-b})^2 = a + b \pm 2\sqrt{a^2 - b^2} + a - b$
$= 2a \pm 2\sqrt{a^2 - b^2}$;
$ax^2 - \frac{1}{4}x^4 = x^2(a - \frac{1}{4}x^2) = \{2a \pm 2\sqrt{a^2 - b^2}\}\{a - (\frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2 - b^2})\}$
$= \{2a \pm 2\sqrt{a^2 - b^2}\}\{\frac{1}{2}a \mp \frac{1}{2}\sqrt{a^2 - b^2}\} = \{a \pm \sqrt{a^2 - b^2})(a \mp \sqrt{a^2 - b^2})$
$= a^2 - (a^2 - b^2) = b^2$

191. $ax^3 + (ay + az + 2cy)x^2 + (by^2 + 2cy^2 + 2cyz)x + (by^3 + by^2z)$ $\div x + (y + z)$

$$\begin{array}{r|llll} 1 & a & +(ay+az+2cy) & +(by^2+2cy^2+2cyz) & +(by^3+by^2z) \\ -(y+z) & & -(ay+az) & -(2cy^2+2cyz) & -(by^3+by^2z) \\ \hline & a & +2cy & +by^2 & \end{array}$$

$\therefore$ quotient $= ax^2 + 2cyx + by^2$

192. $(x^n)^2 - (1^n)^2 \div (x^n - 1^n) = x^n + 1^n = x^n + 1$

193. $1 - 1 + 1 - x + 2x - 3 + 5x + 2 + 4 - 5x = 4 + x$

194. $a(b^2 + 2bc + c^2) + b(c^2 + 2ca + a^2) + c(a^2 + 2ab + b^2) - \{(a^2 - ab - ac + bc)(b + c) + (b^2 - bc - ab + ac)(a + c) + (c^2 - ac - bc + ab)(a + b)\} = ab^2 + ac^2 + bc^2 + ba^2 + ca^2 + cb^2 + 6abc - a^2b - a^2c - b^2a - b^2c - c^2a - c^2b + 6abc = 12abc$

195. $\{(b + c - a) + (c + a - b) + (a + b - c)\}x + \{(c + a - b) + (a + b - c) + (b + c - a)\}y + \{(a + b - c) + (b + c - a) + (c + a - b)\}z = (a + b + c)x + (a + b + c)y + (a + b + c)z = (a + b + c)(x + y + z)$

196. $(x + 2y)^3 \times (x - 2y)^3 = (x^2 - 4y^2)^3 = x^6 - 12x^4y^2 + 48x^2y^4 - 64y^6$

$(a + b\sqrt{-1})(a - b\sqrt{-1}) = a^2 - b^2(-1) = a^2 + b^2$

197. $\{(a + b + c)(a + b - c)\}\{(c - a + b)(c + a - b)\}$ $= \{(a + b)^2 - c^2\}\{c^2 - (a - b)^2\} = c^2\{(a + b)^2 + (a - b)^2\} - (a^2 - b^2)^2 - c^4$ $= 2a^2c^2 + 2b^2c^2 - a^4 + 2a^2b^2 - b^4 - c^4 = 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4$

$(x + 1 + x^{-1})(x - 1 + x^{-1}) = \{(x + x^{-1}) + 1\}\{(x + x^{-1}) - 1\}$ $= (x + x^{-1})^2 - 1 = x^2 + 2 + x^{-2} - 1 = x^2 + 1 + x^{-2}$

198. $(2x^4 - 3x^3y + 4x^2y^2 - 5xy^3 + 6y^4) \div 6x^2y^2$ $= \frac{1}{3}x^2y^{-2} - \frac{1}{2}xy^{-1} + \frac{2}{3} - \frac{5}{6}x^{-1}y + x^{-2}y^2$

$$(x^4 + 4x + 3) \div (x^2 + 2x + 1) = \frac{(x^2 + 2x + 1)(x^2 - 2x + 3)}{x^2 + 2x + 1} = x^2 - 2x + 3$$

199. $(8x - y^3) \div \left(x^{\frac{1}{3}} - \frac{y}{2}\right) = \left\{8\left(x - \frac{y}{2}\right)^3\right\} \div \left(x^{\frac{1}{3}} - \frac{y}{2}\right)$

$$= 8\left\{\frac{x - \left(\frac{y}{2}\right)^3}{x^{\frac{1}{3}} - \frac{y}{2}}\right\} = 8(x^{\frac{2}{3}} + \tfrac{1}{2}x^{\frac{1}{3}}y + \tfrac{1}{4}y^2) = 8x^{\frac{2}{3}} + 4x^{\frac{1}{3}}y + 2y^2$$

$(x^3 - apx^2 + a^2px - a^3) \div (x - a) = \{(x^3 - a^3) - apx(x - a)\} \div (x - a)$
$= (x^2 + ax + a^2) - apx = x^2 + (1 - p)ax + a^2$

200. (I) $(x^2 - 3x - 4) = (x + 1)(x - 4)$, and $(x^2 - 2x - 8) = (x + 2)(x - 4)$; $(x^2 + x - 20) = (x - 4)(x + 5)$ $\therefore$ $G.C.M. = x - 4$

(II) $3x^3 + 4x^2 - 3x - 4 = x^2(3x + 4) - (3x + 4) = (x^2 - 1)(3x + 4)$
$2x^4 - 7x^2 + 5 = (2x^2 - 5)x^2 - (2x^2 - 5) = (2x^2 - 5)(x^2 - 1)$
$\therefore$ $G.C.M. = x^2 - 1$

(III) Let p be the $G.C.M.$ of m and n; then the $G.C.M.$ of $(x^m + a^m)$, and $(x^n + a^n) = x^p + a^p$, and of $(x^m - a^m)$, and $x^n - a^n = x^p - a^p$ $\therefore$ required $G.C.M. = (x^p + a^p)(x^p - a^p) = x^{2p} - a^{2p}$

201. (I) $l.c.m.$ of $(x - 2a)(x + a)$, $x^2(x + a)$, and $a(x + a)(x - a)$
$= ax^2(x - 2a)(x + a)(x - a) = ax^5 - 2a^2x^4 - a^3x^3 + 2a^4x^2$

(II) $x^3 - x^2y - a^2x + a^2y = (x^2 - a^2)(x - y)$; $x^3 + ax^2 - xy^2 - ay^2 = (x + a)(x^2 - y^2)$ $\therefore$ $l.c.m. = (x^2 - a^2)(x^2 - y^2) = x^4 - x^2y^2 - a^2x^2 + a^2y^2$

202. $$\frac{(a+b-c+d)(a+b-c-d)}{(a+b-c-d)(a+b+c+d)} + \frac{(b+c-a+d)(b+c-a-d)}{(b+c+a+d)(b+c-a-d)}$$
$$+ \frac{(c+a-b+d)(c+a-b-d)}{(c+a+b+d)(c+a-b-d)}$$
$$= \frac{a+b-c+d}{a+b+c+d} + \frac{b+c-a+d}{a+b+c+d} + \frac{a+c-b+d}{a+b+c+d}$$
$$= \frac{a+b-c+d+b+c-a+d+a+c-b+d}{a+b+c+d} = \frac{a+b+c+3d}{a+b+c+d}$$
$$= 1 + \frac{2d}{a+b+c+d}$$

203. $$\frac{x^2+2xy+y^2-z^2}{x^2-y^2+2yz-z^2} = \frac{(x+y)^2-z^2}{x^2-(y-z)^2} = \frac{(x+y+z)(x+y-z)}{(x+y-z)(x-y+z)}$$
$$= \frac{x+y+z}{x-y+z}$$

204. $$\frac{a^2(a+b)}{b(a^2-b^2)} - \frac{a(a-b)}{b(a+b)} - \frac{2ab}{a^2-b^2} = \frac{a^3+a^2b-a(a-b)^2-2ab^2}{b(a^2-b^2)}$$
$$= \frac{a^3+a^2b-a^3+2a^2b-ab^2-2ab^2}{b(a^2-b^2)} = \frac{3a^2b-3ab^2}{b(a^2-b^2)} = \frac{3ab(a-b)}{b(a-b)(a+b)}$$
$$= \frac{3a}{a+b}$$

205. $\left(\dfrac{a^2 - ax + ax}{a - x} \times \dfrac{a^2 + ax - ax}{a + x}\right) \div \dfrac{(a+x)^2 + (a-x)^2}{a^2 - x^2}$

$= \left(\dfrac{a^2}{a-x} \times \dfrac{a^2}{a+x}\right) \div \dfrac{2a^2 + 2x^2}{a^2 - x^2} = \dfrac{a^4}{a^2 - x^2} \div \dfrac{a^2 - x^2}{2(a^2+x^2)} = \dfrac{a^4}{2(a^2+x^2)}$

206. $\dfrac{\dfrac{4ab}{a+b} + 2a}{\dfrac{4ab}{a+b} - 2a} + \dfrac{\dfrac{4ab}{a+b} + 2b}{\dfrac{4ab}{a+b} - 2b} = \dfrac{6ab + 2a^2}{2ab - 2a^2} + \dfrac{6ab + 2b^2}{2ab - 2b^2}$;

dividing numerator and denominator of 1st by $2a$, and of 2nd by $2b$,

we get $\dfrac{3b + a}{b - a} + \dfrac{3a + b}{a - b} = \dfrac{3b + a}{b - a} - \dfrac{3a + b}{b - a} = \dfrac{2b - 2a}{b - a} = 2$

207. (I) $\sqrt{x^4 - 4x^3 + 4x^2 - 4x^2 + 8x + 4}$

$= \sqrt{(x^4 - 4x^3 + 4x^2) - 4(x^2 - 2x) + 4} = x^2 - 2x - 2$

(II) $\sqrt{x^{4n}\left\{4 - 4\dfrac{x^n}{3} + \left(\dfrac{x^n}{3}\right)^2\right\}} = x^{2n}\left(2 - \dfrac{x^n}{3}\right) = 2x^{2n} - \dfrac{x^{3n}}{3}$

(III) $\left(\dfrac{a^2}{b^2} - \dfrac{2ab}{bc} - \dfrac{2ac}{ab} + \dfrac{b^2}{c^2} + \dfrac{2bc}{ac} + \dfrac{c^2}{a^2}\right)^{\frac{1}{2}} = \dfrac{a}{b} - \dfrac{b}{c} - \dfrac{c}{a}$

208. The square of which $a^2x^2 + bx$ are the first and second terms, is $a^2x^2 + bx + \dfrac{b^2}{4a^2}$ $\therefore$ in order that $a^2x^2 + bx + bc + b^2$ may be a perfect square, we must have $bc + b^2 = \dfrac{b^2}{4a^2}$ $\therefore \dfrac{c}{b} + 1 = \dfrac{1}{4a^2}$, and $\therefore \dfrac{1}{4a^2} - \dfrac{c}{b} = 1$

209. (I) $mnx + amn = n^2x + am^2$ $\therefore$ $mnx - n^2x = am^2 - amn$, that is $(mn - n^2)x = am^2 - amn$ $\therefore$ $x = \dfrac{am(m-n)}{n(m-n)} = \dfrac{am}{n}$

(II) $2x^2 - 13x = -6$, whence $x = 6$ or $\frac{1}{2}$

210. (I) $\dfrac{7x + 1}{13 - 6x} = \dfrac{400}{3}\left(\dfrac{x - \frac{1}{2}}{x - \frac{2}{3}}\right) = \dfrac{400x - 200}{3x - 2}$

whence $2421x^2 - 6411x = -2598$, or $807x^2 - 2137x = -866$;

$x^2 - \dfrac{2137}{807}x + \left(\dfrac{2137}{1614}\right)^2 - \dfrac{4566769 - 2795448}{(1614)^2} = \dfrac{1771321}{(1614)^2}$

$\therefore x - \frac{2137}{1614} = \pm \frac{\sqrt{1771321}}{1614} = \frac{\pm 1330\cdot9}{1614} \therefore x = \frac{2137 \pm 1330\cdot9}{1614} = +2\cdot14$ or $-0\cdot49$

(II) $x^2 - 2(a+b)x + (a+b)^2 = 4(a^2 - 2ab + b^2) = 4(a-b)^2$ $\therefore x - (a+b) = \pm 2(a-b)$, whence $x = 3a - b$, or $3b - a$

211. $cx - acy = abx + by \therefore x(c - ab) = y(ac + b)$ $\therefore y = \frac{x(c-ab)}{ac+b}$. But $x - ay = b \therefore x - \frac{ax(c-ab)}{ac+b} = b$ $\therefore \frac{acx + bx - acx + a^2bx}{ac+b} = \frac{bx + a^2bx}{ac+b} = x\cdot\frac{b + a^2b}{ac+b} = b$ $\therefore x = \frac{ac+b}{1+a^2}$, and $y = \frac{c - ab}{1+a^2}$

(II) $x^2 + 2xy + y^2 = 49$, and $x^2 + xy + y^2 = 37 \therefore xy = 12$

$$\frac{x^2 - 2xy + y^2 = 1}{}$$

$\therefore x + y = \pm 7$, and $x - y = \pm 1 \therefore 2x = \pm 8$, and $x = \pm 4$ or ± 3, and $y = \pm 3$ or ± 4

212. Subtracting the second of the given equations from the first, we have $y(z - x) = a^2 - c^2$; to which adding the third equation, we have $2yz = 2a^2 \therefore yz = a^2 \therefore xz = b^2$, and $xy = c^2$ $\therefore \frac{yz}{xz} = \frac{y}{x} = \frac{a^2}{b^2} \therefore y = \frac{a^2x}{b^2} \therefore xy = x\cdot\frac{a^2x}{b^2} = c^2 \therefore x^2 = \frac{b^2c^2}{a^2} \therefore x = \pm\frac{bc}{a}$ whence also $y = \pm\frac{ac}{b}$, and $z = \pm\frac{ab}{c}$

213. Let $x = A$'s age, $y = B$'s age, and $z = C$'s age; then $y - x = 2(z - y)$; $x + y = \frac{3z}{2}$, and $x + y - 12 = \frac{4}{3}(z - 6)$; $3y - x = 2z$; $2x + 2y = 3z$; $3x + 3y - 4z = 12$; $6x + 6y - 9z = 0$, and $6x + 6y - 8z = 24 \therefore z = 24$; $y = 21$, and $x = 15$

214. $S_{12} = \{2a + (n-1)d\}\frac{n}{2} = \{3 + (12-1)\frac{3}{2}\}6 = (3 + 11 \times \frac{3}{2})6$ $= (3 + \frac{33}{2}) \times 6 = 117$

S_n of $1\frac{2}{3} + 2\frac{4}{9} + 3\frac{8}{27} = S_n$ of $1 + 2 + 3 + \&c. \dots + S_n$ of $\frac{2}{3} + \frac{4}{9} + \frac{8}{27}$

$= \{2 + (n - 1)1\}\dfrac{n}{2} = (2 + n - 1)\dfrac{n}{2} = \dfrac{n(n + 1)}{2} + \dfrac{\frac{2}{3}\{1 - (\frac{2}{3})^n\}}{1 - \frac{2}{3}}$
$= \dfrac{n(n + 1)}{2} + 2 - 2(\tfrac{2}{3})^n = \tfrac{1}{2}\{n(n + 1) + 4 - 4(\tfrac{2}{3})^n\}$

S_∞ of $\sqrt{2} + \frac{2}{3}\sqrt{3} + \frac{2}{3}\sqrt{2} = \dfrac{\sqrt{2}}{1 - \sqrt{\frac{2}{3}}} = \dfrac{\sqrt{2} + \frac{2}{3}\sqrt{3}}{1 - \frac{2}{3}} = \dfrac{\sqrt{2} + \frac{2}{3}\sqrt{3}}{\frac{1}{3}} = 3\sqrt{2} + 2\sqrt{3}$

215. $a_1 \cdot a_2 \cdot a_3 \cdots a_t = a^{t^2}$ $\therefore$ $a_1 \cdot a_2 \cdot a_3 \cdots a_{p-1} = a_1^{(p-1)^2}$ $a_1 \cdot a_2 \cdot a_3 \cdots a_p = a_1^{p^2}$ $\therefore$ $a_p = a_1^{p^2} \div a_1^{(p-1)^2} = a_1^{2p-1}$ $\therefore$ a_r is formed from a_{r-1} by multiplying it by a_1^2 $\therefore$ $a_1 + a_2 + a_3 +$ &c., is a Geom. series having a_1 for first term, and a_1^2 for common ratio. Then $S_n = a_1 \cdot \dfrac{(a_1^2)^n - 1}{a_1^2 - 1} = a_1 \cdot \dfrac{a_1^{2n} - 1}{a_1^2 - 1}$

216. Squaring each side and transposing, we get $x^4 - 20x^3 + 94x^2 + 60x + 9 = 0$; extracting the square root of each side, we have $x^2 - 10x - 3 = 0$ $\therefore$ $x^2 - 10x + 25 = 28$; $x - 5 = \pm 2\sqrt{7}$ $\therefore$ $x = 5 \pm 2\sqrt{7}$

217. Multiplying, we have $-30x^4 + 46x^3 + 7x^2 - 23x + 4 = 4$
$\therefore x(30x^3 - 46x^2 - 7x + 23) = 0$
$\therefore x\{30x^3 - 30x^2 - 16x^2 + 16x - 23x + 23\} = 0$
$\therefore x\{30x^2(x - 1) - 16x(x - 1) - 23(x - 1)\} = 0$
$\therefore x(x - 1)(30x^2 - 16x - 23) = 0$ $\therefore$ $x = 0$. Also $x - 1 = 0$ $\therefore$ $x = 1$
Also $30x^2 - 16x = 23$, whence $x^2 - \frac{8}{15}x + \frac{16}{225} = \frac{377}{450}$
$\therefore x = \frac{1}{15}(4 \pm \frac{1}{2}\sqrt{754})$

218. The given series is double, i. e. is equal to the *A* series $1 + 2 + 3 + 4 + 5 +$ &c., $+$ the *G* series $1 - 2 + 4 - 8 + 16 -$ &c. Then sum of *A* series as follows :—

$$S_{4n} = \{2 + (4n - 1)\}\frac{4n}{2} = (4n + 1)2n$$

$$S_{4n+1} = \{2 + (4n + 1 - 1)\}\frac{4n + 1}{2} = (2n + 1)(4n + 1)$$

$$S_{4n+2} = \{2 + (4n + 2 - 1)\}\frac{4n + 2}{2} = (4n + 3)(2n + 1)$$

$$S_{4n+3} = \{2 + (4n + 3 - 1)\}\frac{4n + 3}{2} = 2(n + 1)(4n + 3)$$

Also sum of G series as follows :—

$$S_{4n} = \frac{(-2)^{4n} - 1}{-2 - 1} = \frac{16^n - 1}{-3} = \frac{1}{3}(1 - 16^n)$$

$$S_{4n+1} = \frac{(-2)^{4n+1} - 1}{-3} = \frac{1}{3}\{1 - (-2)^{4n+1}\}$$

$$S_{4n+2} = \frac{(-2)^{4n+2} - 1}{-3} = \frac{1}{3}(1 - 4^{2n+1})$$

$$S_{4n+3} = \frac{(-3)^{4n+3} - 1}{-3} = \frac{1}{3}\{1 - (-2)^{4n+3}\}$$

$\therefore$ of given series $S_{4n} = 2n(4n + 1) + \frac{1}{3}(1 - 16^n)$

$$S_{4n+1} = (2n + 1)(4n + 1) + \tfrac{1}{3}\{1 - (-2)^{4n+1}\}$$

$$S_{4n+2} = (4n + 3)(2n + 1) + \tfrac{1}{3}(1 - 4^{2n+1})$$

$$S_{4n+3} = 2(n + 1)(4n + 3) + \tfrac{1}{3}\{1 - (-2)^{4n+2}\}$$

219. Let x = number in width, and y = number in the length; then xy = whole number in the bunch. Also, since $y > 10$ but < 20, y = a number of two digits $\therefore$ when x is written to the left of y it must occupy the third or hundreds place $\therefore$ $100x + y$ = the number in scale of 10.

Also since $x < 10$, it consists of but one digit, therefore when written to the left of y, the number will be represented by $10y + x$ which $\therefore$ = number in scale of 10

Again in similar rectangles the perimeters are as the corresponding sides, and whole perimeter of first bunch = $2(x + y)$, and of second bunch xy $\therefore$ $2(x + y) : xy :: x : \dfrac{x^2y}{2(x + y)}$ = width of 2nd bunch, and $2(x + y) : xy :: y : \dfrac{xy^2}{2(x + y)}$ = length of 2nd bunch $\therefore$ whole number of matches in the second bunch

$$= \frac{x^2y}{2(x + y)} \times \frac{xy^2}{2(x + y)} = \frac{x^3y^3}{4(x + y)^2}$$

Then from first condition $100x + y : xy :: a : 2$ (I)

" second " $10y + x : xy :: a - 10 : 4$ (II)

" third " $\dfrac{x^3y^3}{4(x + y)^2} = 4xy$ (III)

From (II) $20y + 2x : xy :: a - 10 : 2$ $\therefore$ $20y + 2x + 5y : xy :: a : 2$
$\therefore$ $20y + 2x + 5xy : xy :: 100x + y : xy$
$\therefore$ $20y + 2x + 5xy = 100x + y$ $\therefore$ $5xy = 98x - 19y$ (IV)
Also from (III) $x^2y^2 = 16(x+y)^2$ $\therefore$ $xy = 4(x+y)$ $\therefore$ $5xy = 20(x+y)$
Substituting this in (IV), we have $20x + 2y = 98x - 19y$, whence $2x = y$.

Again substituting this in (III), we have $x^2y^2 = 16(x+y)^2$, that is $4x^4 = 16(x+2x)^2$ $\therefore$ $2x^2 = 4 \times (3x)$ $\therefore$ $2x^2 = 12x$, or $x = 6$ $\therefore$ $y = 12$; and $xy = 6 \times 12 = 72$ = number of matches in the bunch.

220. Since the conditions giving the equations (I) and (II) remain the same, these equations and $\therefore$ also (IV) which is derived from them independently of (III), remain the same.

$\therefore$ we have but to solve in *positive** integers the equation $5xy = 98x - 19y$, remembering that $x < 10$, and $y > 10$ but < 20

$$5xy = 98x - 19y$$
$$\therefore\ 5xy + 19y = 98x$$
$$\therefore\ y = \frac{98x}{5x+19}$$
$$\therefore\ 5y = \frac{490x}{5x+19} = 98 - \frac{1862}{5x+19}$$

Now since y is an integer, $\dfrac{1862}{5x+19}$ is also an integer.

And since x is integral, $5x + 19$ must equal an integral divisor of 1862, and further since x is finite, positive and less than 10, $5x + 19$ will be > 19 but < 69 and will end in 9 or 4 according as x is even or odd.

Now the only divisor of 1862 fulfilling these conditions is 49

$$\therefore\ 5x + 19 = 49$$
$$\therefore\ y = 6$$
$$y = \frac{98x}{5x+19} = 12$$
$$\therefore\ xy = 72$$

*They must be positive from the nature of the problem.

221. Let x = rate per hour of the express down; y = rate per hour of accommodation down, and d = distance from Stratford to Toronto. Then $\frac{d}{x}$ hours = time down by express, and $\frac{d}{y}$ = time down by accommodation. Also $\frac{d}{x}$ = cents per mile in express fare, and $\frac{d}{x} \times d = \frac{d^2}{x}$ = whole fare by express. $x - \frac{d}{x}$ = rate of expressing going up $\therefore \frac{d}{x - \frac{d}{x}}$ = hours on road going back

But if the fares had varied as the velocities; then fare at x : fare at y :: $x : y$ $\therefore$ fare at x − fare at y : fare at x :: $x - y : x$

But in this case, fare at x − fare at $y = d$ cents, and since fare by express to Toronto remains the same, $d : \frac{d^2}{x} :: x - y : x$ (I)

Also fare at x : fare at $\left(x - \frac{d}{x}\right) :: x : x - \frac{d}{x}$ $\therefore$ fare at x − fare at $\left(x - \frac{d}{x}\right)$: fare at x :: $\frac{d}{x} : x$

But fare at x − fare at $\left(x - \frac{d}{x}\right) = x - \frac{d}{x}$ cents

$\therefore x - \frac{d}{x} : \frac{d^2}{x} :: \frac{d}{x} : x$ (II)

Using the formulas now found in expressing the remaining statements in the problem, we obtain $\frac{d}{y} = \frac{1}{2}\left(\frac{d}{x}\right)^2$ (III);

Then from (III) $dy = 2x^2$ (IV)

from (I) $x^2 = d(x - y) = dx - 2x^2$, by (IV) $\therefore d = 3x$ (V)

from (II) $x^2(x^2 - d) = d^3$ $\therefore$ by (V) $x^2(x^2 - 3x) = 27x^3$

$\therefore x^2 - 3x = 27x$ $\therefore x - 3 = 27$ $\therefore x = 30$ $\therefore d = 3x = 90$ miles = distance from Toronto to Stratford; and $\frac{d^2}{x} = \frac{90 \times 90}{30}$ = 270 cents = \$2·70 = fare from Toronto to Stratford.

222. $x^2\sqrt{x^2+25}(x^2+9)(\sqrt{x^2+25}-1) - 45\sqrt{x^2+25} = 5(x^2+45)$

$\therefore$ (I) $x^2\sqrt{x^2+25}(x^2+9)(\sqrt{x^2+25}-1) = 5(x^2+25+9\sqrt{x^2+25}+20)$

$= 5\{(\sqrt{x^2+25})(\sqrt{x^2+25}) + 9\sqrt{x^2+25} + 20\}$

$= 5\{(\sqrt{x^2+25})(\sqrt{x^2+25}) + 5\sqrt{x^2+25} + 4\sqrt{x^2+25} + 20\}$

$= 5(\sqrt{x^2+25}+5)(\sqrt{x^2+25}+4)$ (II)

$$\left.\begin{array}{l}\text{But } (\sqrt{x^2+25}+5)(\sqrt{x^2+25}-5) = x^2 \\ \text{And } (\sqrt{x^2+25}+4)(\sqrt{x^2+25}-4) = x^2+9\end{array}\right\} \therefore \text{ from (I) and (II)}$$

$(\sqrt{x^2+25}+5)(\sqrt{x^2+25}-5)(\sqrt{x^2+25})(\sqrt{x^2+25}+4)(\sqrt{x^2+25}-4)$
$(\sqrt{x^2+25}-1) = 5(\sqrt{x^2+25}+5)(\sqrt{x^2+25}+4)$

$\therefore (\sqrt{x^2+25}-5)(\sqrt{x^2+25})(\sqrt{x^2+25}-4)(\sqrt{x^2+25}-1) = 5$ (III)

$\therefore (x^2+25-5\sqrt{x^2+25})(x^2+25-5\sqrt{x^2+25}+4) = 5$

$\therefore (x^2+25-5\sqrt{x^2+25})^2 + 4\,(x^2+25-5\sqrt{x^2+25}) = 5$

$\therefore (x^2+25-5\sqrt{x^2+25})^2 + 4(x^2+25-\sqrt{x^2+25}) + 4 = 9$

$\therefore x^2+25-5\sqrt{x^2+25} = -5$ or 1

$\therefore (x^2+25) - 5\sqrt{x^2+25} + \frac{25}{4} = \frac{5}{4}$ or $\frac{29}{4}$

$\therefore \sqrt{x^2+25} = \dfrac{5 \pm \sqrt{5}}{2}$ or $\dfrac{5 \pm \sqrt{29}}{2}$

$\therefore x^2+25 = \frac{1}{4}(30 \pm 10\sqrt{5})$ or $\frac{1}{4}(54 \pm 10\sqrt{29})$

Whence $x = \frac{1}{2}(\sqrt{\pm 10\sqrt{5} - 70})$ or $\frac{1}{2}(\sqrt{\pm 10\sqrt{29} - 46})$

Also $\sqrt{x^2+25}+5 = 0$, whence $\sqrt{x^2+25} = -5$, or $x^2+25 = 25$ $\therefore x = 0$
$\sqrt{x^2+25}+4 = 0$, whence $\sqrt{x^2+25} = -4$, or $x^2+25 = 16$ $\therefore x^2 = -9$,
or $x = \pm 3\sqrt{-1}$

223. Let x = number of yards dug at \$1·25; then $100 - x$ = number of yards dug at \$0·75 $\therefore$ $1{\cdot}25x = 50 = {\cdot}75(100 - x)$. Therefore, we have two independent equations containing only one unknown quantity, and any solution obtained from one equation is inconsistent with the other; consequently the problem is impossible.

224. Let x = length of one side of rectangle and y = other; then xy = area, and $2(x + y)$ = perimeter of the rectangle; and xy = area and $4\sqrt{xy}$ = perimeter of the square.

(I) $\therefore xy = 4m\sqrt{xy}$ (I), and $xy = 2n(x+y)$ (II). From (I) $\sqrt{xy} = 4m \therefore xy = 16m^2$ (III), substitute this in (II), and we get $16m^2 = 2n(x+y) \therefore 8m^2 = n(x+y)$ (IV). Squaring (IV), we have $64m^4 = n^2(x+y)^2$; multiplying (III) by $4n^2$, we have $64m^2n^2 = 4n^2xy$ $\therefore$ by subtraction $64m^2(m^2-n^2) = n^2(x-y)^2 \therefore \pm 8m\sqrt{m^2-n^2} = n(x-y)$ (V)

Adding (IV) and (V) and reducing, we get $x = \frac{4m}{n}(m \pm \sqrt{m^2-n^2})$

Taking (V) from (IV) and reducing, we get $y = \frac{4m}{n}(m \mp \sqrt{m^2-n^2})$

(II) When the perimeters are equal; then taking x and y as before, we have $2(x+y)$ = perimeter of the square, and $\frac{(x+y)^2}{4}$ = its area; $\frac{(x+y)^2}{4} = 2m(x+y)$ (I); $xy = 2n(x+y)$ (II). From (I) $x+y = 8m$ (III), substitute this in (II), and $xy = 16mn$ (IV) Square (III), subtract 4 times (IV) and then take the square root, and we have $x - y = \pm 8\sqrt{m^2 - mn}$ (V) Adding (III) and (V) and reducing, $x = 4(m \pm \sqrt{m^2 - mn})$ Subtracting (V) from (III) and reducing, $y = 4(m \mp \sqrt{m^2 - mn})$

225. Let x = age of younger at first trial, and y = age of elder. Let r = ratio of throw to age at first trial, and $r_{,}$ = ratio of gain of one to age of the other at second throw $\therefore$ first throw of younger $= rx$, and first throw of elder $= ry$; gain of younger $= r_{,}(y+1)$; gain of elder $= r_{,}(x+1)$; second throw of younger $= rx + r_{,}(y+1)$; second throw of elder $= ry + r_{,}(x+1)$. Also *H.* mean of their ages at latter trial $= \frac{2(x+1)(y+1)}{x+y+2}$; *A.* mean of first throws $= \frac{r(x+y)}{2}$, and *A.* mean of 2nd throws $= \frac{r(x+y) + r_{,}(x+y+2)}{2}$ $\therefore$ difference of *A.* means $= \frac{r_{,}(x+y+2)}{2}$

Longest throw = second throw of the elder $= ry + r_{,}(x+1)$; value of ratios compounded of ratio of throw to age and gain

to age of other $= rr_{,}$; value of ratio formed by multiplying antecedent of this compound ratio by $\frac{1}{4}$ product of ages at second trial $= \frac{1}{4}rr_{,}(x+1)(y+1)$; value of the ratio of which this is the duplicate $= \frac{1}{2}\sqrt{rr_{,}(x+1)(y+1)}$; value of the ratio compounded of the ratio of throw to age of one with gain of one to age of other $= \dfrac{r}{r_{,}}$

Then using the values thus expressed in stating the problem, we have the four equations:—

$ry - rx = 24$; or $r(y-x) = 24$ (I)

$\{ry + r_{,}(x+1)\} - \{rx + r_{,}(y+1)\} = 25$; or $(r - r_{,})(y-x) = 22$ (II)

$\dfrac{ry + r_{,}(x+1)}{\dfrac{r_{,}(x+y+2)}{2}} = \dfrac{2(x+1)(y+1)}{x+y+2}$; or $r = r_{,}(x+1)$ (III)

$\frac{1}{2}\sqrt{rr_{,}(x+1)(y+1)} = \dfrac{r}{r_{,}}$; or $(x+1)(y+1)r_{,}^3 = 4r$ (IV)

Then (I) − (II) gives $r_{,}(y-x) = 2$ (V); substituting (III) in (II) $r_{,}x(y-x) = 22$ (VI); dividing (IV) by (V), we have $x = 11$ = age of younger at first throw (VII). Substituting (VII) in (III) $r = 12r_{,}$, and in (V) $r_{,}(y-11) = 2$; also substituting (III) in (IV) and reducing, $r_{,}^2(y+1) = 4$ (VIII). But $r_{,}(y-11) = 2$ $\therefore r_{,}^2(y-11) = 2r_{,}$; subtracting this from (VIII), we have $12r_{,}^2 = 4 - 2r_{,}$ $\therefore 6r_{,}^2 + r_{,} = 2$ $\therefore r_{,}^2 + \frac{1}{6}r_{,} + \frac{1}{144} = \frac{1}{3} + \frac{1}{144} = \frac{49}{144}$ $\therefore r_{,} = \pm\frac{7}{12} - \frac{1}{12} = \frac{1}{2}$. But $r = 12r_{,}$ $\therefore r = 6$, and since $r_{,}(y-11) = 2$; $\frac{1}{2}(y-11) = 2$ $\therefore y - 11 = 4$, or $y = 15$ = age of elder $\therefore$ throws at first trial $= 11 \times 6 = 66$, and $15 \times 6 = 90$; and throws at second trial $= 66 + \frac{1}{2}(15+1) = 74$, and $90 + \frac{1}{2}(11+1) = 96$

THE END.

John M. [illegible]

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